Section 9.1, Infinite Sequences

Homework: 9.1 #1–35 odds

In this chapter, we will shift our focus from derivatives and integrals to sequences and series. However, we will be using derivatives and integrals occasionally to find some properties of both sequences and series.

1 Sequences

An **infinite sequence** is an ordered arrangement of an infinite number of real numbers. Numbers may be repeated, and do not necessarily need to be in increasing or decreasing order.

There are several different ways that sequences are denoted. Some are:

$$a_1, a_2, a_3, a_4, \dots$$
$$\{a_n\}_{n=1}^{\infty}$$
$$\{a_n\}$$

There are three primary ways that sequences can be defined, or listed. One way is to list enough numbers to see the pattern, such as

$$1, 7, 13, 19, 25, \dots$$

Another way is by **iteration**, or an **explicit formula**, such as

$$a_n = 6n - 5 \quad , n \geq 1$$

The third way is by **recursion** or an implicit formula, such as

$$a_1 = 1$$
$$a_n = a_{n-1} + 6$$

2 Convergence

One of our primary goals for this section will be to see if limits of these sequences exist, and what that limit is.

We say that a sequence $\{a_n\}$ **converges** to L, or $\lim_{n\to\infty} a_n = L$ if for all $\varepsilon > 0$, there exists a corresponding positive N such that for all integers $n \geq N$, then $|a_n - L| < \varepsilon$.

If a sequence fails to converge to a finite L, then we say that it **diverges**.

Properties of limits of Sequences
If k is a constant and $\{a_n\}$ and $\{b_n\}$ are convergent sequences.

1. $\lim\limits_{n\to\infty} k = k$
2. $\lim\limits_{n\to\infty} ka_n = k \lim\limits_{n\to\infty} a_n$
3. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \pm \left(\lim\limits_{n\to\infty} b_n\right)$
4. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \cdot \left(\lim\limits_{n\to\infty} b_n\right)$

5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

Another helpful property that we can use is that if $f(x)$ is a continuous function and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$ where the n in the second limit represents only positive integer values. This means that we can use L'Hôpital's Rule, even though the derivative doesn't exist when x can only be integer values.

Examples
Determine whether the sequence converges or diverges, and, if it converges, find $\lim_{n\to\infty} a_n$.

1. $a_n = \frac{n}{2n-1}$ Also Write the first 4 terms of this sequence.

   The first 4 terms are $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}$. This sequence converges to $1/2$ since $\frac{n}{2n-1}$ converges to $1/2$.

2. $a_n = \frac{5n^2 - 3n + 1}{2n^2 + 7}$

   This converges to $5/2$.

3. $a_n = \frac{\ln(1/n)}{\sqrt{2n}}$

   By L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(1/x)}{\sqrt{2x}} &= \lim_{x\to\infty} \frac{-\ln x}{(2x)^{1/2}} \\
&= \lim_{x\to\infty} \frac{-1/x}{(2x)^{-1/2}} \\
&= \lim_{x\to\infty} \frac{-(2x)^{1/2}}{x} = 0
\end{aligned}$$

   so the sequence converges to 0.

4. $a_n = \frac{n^{200}}{e^n}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^{200}}{e^x} &= \lim_{x\to\infty} \frac{200x^{199}}{e^x} \\
&= \lim_{x\to\infty} \frac{200 \cdot 199x^{198}}{e^x} = \dots \\
&= \lim_{x\to\infty} \frac{200!}{e^x} \\
&= \lim_{x\to\infty} \frac{0}{e^x} = 0,
\end{aligned}$$

   so the sequence converges to 0.

The Squeeze Theorem says that if $\{a_n\}$ and $\{c_n\}$ both converge to L and $a_n \leq b_n \leq c_n$ for $n \geq k$, where k is some fixed integer, then $\{b_n\}$ also converges to L.

A direct result of this is that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example
Determine if $\{a_n\}$, where $a_n = e^{-n} \sin n$ converges. If so, find the limit.

Since $-1 \leq \sin n \leq 1$, $-e^{-n} \leq e^{-n} \sin n \leq e^{-n}$. We know that both $-e^{-n}$ and e^{-n} converge to 0, so $e^{-n} \sin n$ also converges to 0.

Monotonic Sequence Theorem
If U is an upper bound for a nondecreasing sequence $\{a_n\}$, then the sequence converges to a limit A that is less than or equal to U. Similarly, if L is a lower bound for a nonincreasing sequence $\{b_n\}$, then the sequence $\{b_n\}$ converges to a limit B that is greater than or equal to L.

Example

Show that $\{a_n\}$ converges if $a_n = \dfrac{n}{n+1}\left(2 - \dfrac{1}{n^2}\right)$.

This sequence is nondecreasing, since both $\frac{n}{n+1}$ and $2 - \frac{1}{n^2}$ are nondecreasing. Since $\frac{n}{n+1} \leq 1$ and $2 - \frac{1}{n^2} \leq 2$, $a_n \leq 1 \cdot 2 = 2$, so we have an upper bound for the sequence. Therefore, by the Monotonic Sequence Theorem, the sequence converges.