Section 8.4, Improper Integrals: Infinite Integrands

Homework: 8.4 #1–33 odds

Consider the definite integral $\int_{-1}^{2} \frac{dx}{x^3}$. Can we calculate it as

$$\int_{-1}^{2} \frac{dx}{x^3} = -\frac{1}{2}x^{-2}\Big|_{-2}^{1} = -\frac{1}{2}\cdot 2^{-2} + \frac{1}{2}\cdot 1 = \frac{3}{8}?$$

No, we can't calculate the definite integral this way! Part of the "fine print" in the Second Fundamental Theorem of Calculus is that the integrand must be a continuous function on the interval of integration (here, $[-1, 2]$). However, $\frac{1}{x^3}$ is discontinuous at $x = 0$.

Definition of Definite Integrals with Infinite Integrands
Let f be continuous on the interval $[a, b)$ and let $\lim_{x\to b^-} |f(x)| = \infty$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided that this limit exists and is finite. In this case, we say that the integral converges. Otherwise, we say that the integral diverges.

An analogous statement holds if $|f(x)| \to \infty$ at the lower end of the interval.

Examples

1. $\int_1^3 \frac{dx}{(x-1)^{4/3}}$
   The integrand has a vertical asymptote at $x = 1$, so:

$$\begin{aligned}\int_1^3 \frac{dx}{(x-1)^{4/3}} &= \lim_{t\to 1^+} \int_t^3 (x-1)^{-4/3}\,dx \\ &= \lim_{t\to 1^+} \big(-3(x-1)^{-1/3}\big)\Big|_t^3 \\ &= \lim_{t\to 1^+} \big(-3\cdot 2^{-1/3} + 3(t-1)^{-1/3}\big),\end{aligned}$$

   which is infinite, so this integral diverges.

2. $\int_0^4 \frac{dx}{\sqrt{4-x}}$
   The integrand has a vertical asymptote at $x = 4$, so

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{4-x}} &= \lim_{t\to 4^-} \int_0^t (4-x)^{-1/2}\,dx \\ &= \lim_{t\to 4^-} \big(-2(4-x)^{1/2}\big)\Big|_0^t \\ &= \lim_{t\to 4^-} \big(-2(t-x)^{1/2} + 2\cdot 4^{1/2}\big) \\ &= 4\end{aligned}$$

3. $\int_0^1 x^{-p}\,dx$ if $p > 0$.

If $p = 1$,

$$\int_0^1 x^{-1}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-1}\,dx$$
$$= \lim_{t\to 0^+} \ln|x|\Big|_t^1 = \lim_{t\to 0^+} (\ln 1 - \ln) = \infty$$

so the integral diverges.

If $p \neq 1$,

$$\int_0^1 x^{-p}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-p}\,dx$$
$$= \lim_{t\to 0^+} \frac{1}{1-p} x^{1-p}\Big|_t^1$$
$$= \lim_{t\to 0^+} \left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p}\right)$$
$$= \begin{cases} \frac{1}{1-p} & \text{if } 0 < p < 1 \\ \infty & \text{if } p > 1 \end{cases}$$

This means that

$$\int_0^1 x^{-p}\,dx = \begin{cases} \frac{1}{1-p} & \text{if } 0 < p < 1 \\ \infty & \text{if } p \geq 1 \end{cases}$$

If f is continuous on $[a, b]$ except at a number c such that $a < c < b$, and suppose that $\lim_{x\to c} |f(x)| = \infty$. Then define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided that both integrals on the right converge. Otherwise, we say that $\int_a^b f(x)\,dx$ diverges.

Examples

1. $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$

   The integrand has a vertical asymptote at $x = 0$.

   $$\int_{-1}^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}$$

   By the last example, this diverges.

2. $\displaystyle\int_{-3}^1 \frac{5}{(x+2)^{3/5}}\,dx$

   This integrand has a vertical asymptote at $x = -2$

   $$\int_{-3}^1 \frac{5}{(x+2)^{3/5}}\,dx = \int_{-3}^{-2} 5(x+2)^{-3/5}\,dx + \int_{-2}^1 5(x+2)^{-3/5}\,dx$$
   $$= \frac{25}{2}(x+2)^{2/5}\Big|_{-3}^{-2} + \frac{25}{2}(x+2)^{2/5}\Big|_{-2}^{1}$$
   $$= \frac{25}{2}\left(0 - (-1)^{2/5}\right) + \frac{25}{2}\left(3^{2/5} - 0\right)$$
   $$= \frac{25(3^{2/5} - 1)}{2}$$