Section 8.3, Improper Integrals: Infinite Limits of Integration

Homework: 8.3 #1–19 odds, 25, 27, 31

In this section and the next, we will look at **improper integrals**, which are definite integrals where either one or both of the limits of integration are infinite, or the integrand is infinite. In this section, we will focus on definite integrals with infinite limits.

Definition of Improper Integrals with Infinite Limits

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx$$
$$\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{a}^{b} f(x)\,dx$$

If the limit exists and have finite values, we say that the improper integral **converges**. Otherwise, we say that the integral **diverges**.

Examples
Evaluate each of the following integrals or show that it diverges:

1. $\displaystyle\int_{-\infty}^{2} e^x\,dx$

$$\begin{aligned}\int_{-\infty}^{2} e^x\,dx &= \lim_{a\to-\infty}\int_{a}^{2} e^x\,dx\\ &= \lim_{a\to-\infty} e^x\Big|_a^2\\ &= \lim_{a\to-\infty}\left(e^2 - e^a\right) = e^2\end{aligned}$$

2. $\displaystyle\int_{1}^{\infty} \frac{dx}{\sqrt{\pi x}}$

$$\begin{aligned}\int_{1}^{\infty} \frac{dx}{\sqrt{\pi x}} &= \lim_{b\to\infty}\int_{1}^{b} (\pi x)^{-1/2}\,dx\\ &= \lim_{b\to\infty}\frac{2}{\pi}(\pi x)^{1/2}\Big|_1^b\\ &= \lim_{b\to\infty}\frac{2}{\pi}\left((b\pi)^{1/2} - \pi^{1/2}\right)\\ &= \infty,\end{aligned}$$

so this integral diverges.

3. $\displaystyle\int_1^\infty \frac{x}{(1+x^2)^2}\,dx$

$$\begin{aligned}\int_1^\infty \frac{x}{(1+x^2)^2}\,dx &= \lim_{b\to\infty}\int_1^b x(1+x^2)^{-2}\,dx \\ &= \lim_{b\to\infty}\left(-\frac{1}{2}(1+x^2)^{-1}\right)\Bigg|_1^b \\ &= \lim_{b\to\infty}\left(-\frac{1}{2}(1+b^2)^{-1}+\frac{1}{2}\cdot\frac{1}{2}\right) \\ &= \frac{1}{4}\end{aligned}$$

4. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$, where p is a real number

We will consider every possible value of p. First, for $p=1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty}\ln x\Big|_1^b \\ &= \lim_{b\to\infty}\ln b = \infty,\end{aligned}$$

so the integral diverges when $p=1$. Now, for $p\neq 1$, the power rule applies:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty}\frac{1}{1-p}x^{1-p}\Big|_1^b \\ &= \lim_{b\to\infty}\left(\frac{1}{1-p}b^{1-p}-\frac{1}{1-p}\right) \\ &= \begin{cases}\infty & \text{if } p<1 \\ \frac{1}{p-1} & \text{if } p>1\end{cases}\end{aligned}$$

This means that for $p\le 1$, the integral diverges, and for $p>1$, it equals $\frac{1}{p-1}$

Definition of Improper Integrals with Two Infinite Limits

If $\int_0^\infty f(x)\,dx$ and $\int_{-\infty}^0 f(x)\,dx$ converge, then $\int_{-\infty}^\infty f(x)\,dx$ converges and

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

Otherwise, $\int_{-\infty}^\infty f(x)\,dx$ diverges.

Example

$$\int_{-\infty}^\infty \frac{dx}{x^2+16}$$

$$\begin{aligned}\int_{-\infty}^\infty \frac{dx}{x^2+16} &= \lim_{a\to-\infty}\int_a^0 \frac{dx}{x^2+16} + \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+16} \\ &= \lim_{a\to-\infty}\frac{1}{4}\tan^{-1}\frac{x}{4}\Big|_a^0 + \lim_{b\to\infty}\frac{1}{4}\tan^{-1}\frac{x}{4}\Big|_0^b \\ &= \lim_{a\to-\infty}\left(0-\frac{1}{4}\tan^{-1}\frac{a}{4}\right) + \lim_{b\to\infty}\left(\frac{1}{4}\tan^{-1}\frac{b}{4}-0\right) \\ &= \frac{1}{4}\cdot\frac{\pi}{2}+\frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{4}\end{aligned}$$

1 Probability Density Functions

A **probability density function** (PDF) is a non-negative function $f(x)$, defined on the real numbers such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

The probability that a random variable X is between a and b is $\int_a^b f(x)\,dx$.

The **mean** of the random variable with the PDF of $f(x)$ is

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx.$$

This is a weighted average of all the value that the random variable can take.

The **variance** is

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu^2) f(x)\,dx.$$

This describes how spread out the distribution is.

Example
A continuous random variable X has an **exponential distribution** with parameter θ for $\theta > 0$ if its PDF is:

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

1. Show that this is a valid PDF.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} \frac{1}{\theta} e^{-x/\theta}\,dx \\ &= 0 + \lim_{b\to\infty} (-e^{-x/\theta})\Big|_0^b \\ &= \lim_{b\to\infty} \big(- e^{-b/\theta} + e^0 \big) = 1 \end{aligned}$$

2. Find the mean μ and variance σ^2 of the distribution.

$$\begin{aligned} \mu &= \int_0^{\infty} \frac{x}{\theta} e^{-x/\theta}\,dx \\ &= -xe^{-x/\theta}\Big|_0^{\infty} + \int_0^{\infty} e^{-x/\theta}\,dx \\ &= 0 + -\theta e^{-x/\theta}\Big|_0^{\infty} \\ &= \theta, \end{aligned}$$

where we used integration by parts.

$$\begin{aligned} \sigma^2 &= \int_0^{\infty} (x-\theta)^2 \frac{1}{\theta} e^{-x/\theta}\,dx \\ &= -(x-\theta)^2 e^{-x/\theta}\Big|_0^{\infty} + \int_0^{\infty} 2(x-\theta) e^{-x/\theta}\,dx \\ &= \theta^2 + (-2)\theta(x-\theta)e^{-x/\theta}\Big|_0^{\infty} + \int_0^{\infty} 2\theta e^{-x/\theta}\,dx \\ &= \theta^2 - 2\theta^2 - 2\theta^2 e^{-x/\theta}\Big|_0^{\infty} \\ &= \theta^2 - 2\theta^2 + 2\theta^2 = \theta^2 \end{aligned}$$

3. If $\theta = 2$, find the probability that X is less than $\ln 4$.

$$
\begin{aligned}
P(X < \ln 4) &= \int_0^{\ln 4} \frac{1}{2} e^{-x/2}\, dx \\
&= -e^{-x/2}\Big|_0^{\ln 4} = -e^{-(\ln 4)/2} + 1 \\
&= \frac{1}{2}
\end{aligned}
$$