Section 8.2, Other Indeterminate Forms

Homework: 8.2 #1–39 odds

In the last section, we will find limits with the indeterminate form $0/0$. In this section, we will look at limits of the form ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, and 1^∞. It turns out that L'Hôpital's Rule works for all of these, too! For all of these forms except ∞/∞, we will need to rearrange the function in the limit first.

In the ∞/∞ case, if $\lim_{x\to c} |f(x)| = \lim_{x\to c} |g(x)| = \infty$ and if $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

where c may represent any real number, c^-, c^+, or $\pm\infty$.

Examples
Calculate each of the following limits:

1. $\lim_{x\to\infty} \frac{x^9}{e^x}$
   This has the form ∞/∞. Applying L'Hôpital's Rule, we get:
   $$\lim_{x\to\infty} \frac{x^9}{e^x} = \lim_{x\to\infty} \frac{9x^8}{e^x} = \cdots = \lim_{x\to\infty} \frac{9!}{e^x} = 0$$

2. $\lim_{x\to\infty} \frac{(\ln x)^2}{2^x}$
   This also has the form ∞/∞, so
   $$\lim_{x\to\infty} \frac{(\ln x)^2}{2^x} = \lim_{x\to\infty} \frac{2(\ln x)}{x2^x \ln 2} = \lim_{x\to\infty} \frac{2 \cdot \frac{1}{x}}{2^x \ln 2 + x2^x(\ln 2)^2} = 0$$

3. $\lim_{x\to 0} 5x \cot x$
   This has the form $0 \cdot \infty$. If we rearrange it, we can get a limit with either the form $0/0$ or ∞/∞, then use L'Hôpital's rule:
   $$\lim_{x\to 0} 5x \cot x = \lim_{x\to 0} \frac{5x}{\tan x} = \lim_{x\to 0} \frac{5}{\sec^2 x} = 5$$

4. $\lim_{x\to\pi/2} \big(\tan x - \sec x\big)$
   This has the form $\infty - \infty$. We can rewrite this as one term using sines and cosines:
   $$\lim_{x\to\pi/2} \big(\tan x - \sec x\big) = \lim_{x\to\pi/2} \frac{\sin x - 1}{\cos x} = \lim_{x\to\pi/2} \frac{\cos x}{-\sin x} = 0$$

5. $\lim_{x\to 0} \big(\cos x\big)^{1/x^2}$
   This has the form 1^∞. Let $y = (\cos x)^{1/x^2}$. Then, $\ln y = \frac{1}{x^2} \ln \cos x$ (this will give us the form $\infty \cdot 0$). Taking the limit for $\ln y$, we get:
   $$\lim_{x\to 0} \frac{\ln \cos x}{x^2} = \lim_{x\to 0} \frac{-\frac{\sin x}{\cos x}}{2x} = \lim_{x\to 0} \frac{-\sin x}{2x \cos x}$$
   $$= \lim_{x\to 0} \frac{-\cos x}{2\cos x - 2x \sin x} = -\frac{1}{2}$$

Since this is the limit of $\ln y$, not y, we need to exponentiate our answer to see that $\lim_{x \to 0} y = e^{-1/2}$.

6. $\lim_{x \to 0^+} x^x$

   This has the form 0^0, so we will use a similar approach to the last example. Let $y = x^x$. Then, $\ln y = x \ln x$, which will give us the form $0 \cdot (-\infty)$. The limit of $\ln y$ is:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} -x = 0,$$

   so $\lim_{x \to 0^+} x^x = \exp\left(\lim_{x \to 0^+} \ln y\right) = e^0 = 1$.

Reminder: Limits with the forms 1^0, 0^∞, ∞^∞, $\infty \cdot \infty$, $\infty + \infty$, $0/\infty$ and $\infty/0$ are not indeterminate. They can all be determined by methods from Calculus I.