Section 7.5, Integration of Rational Functions Using Partial Fractions

Homework: 7.5 #1–43 odds

A **rational function** is the quotient of two polynomial functions. It is called **proper** if the numerator has a lower degree than the denominator.

Partial Fraction Decomposition

The methods we will use here only work for proper rational functions. If your initial fraction is not proper, you will need to carry out long division first, then carry out this method on the "remainder" term.

Our steps will involve first factoring the denominator. Then we rewrite the original fraction as the sum of fractions whose denominators are the individual factors.

For example, to decompose $\frac{x-7}{x^2-x-12}$, we factor the denominator as $(x-4)(x+3)$, then we will try to find constants A and B such that

$$\frac{x-7}{x^2-x-12} = \frac{A}{x-4} + \frac{B}{x+3}$$

(We want A and B to be constants since we need these to be proper rational functions.)

To solve for A and B, multiply both sides of the equation by the denominator $x^2 - x - 12 = (x-4)(x+3)$. Then,

$$x - 7 = A(x+3) + B(x-4) = (A+B)x + (3A - 4B)$$

Since we need the powers of like terms of x to match, we get the equations:

$$\begin{aligned} 1 &= A + B \\ -7 &= 3A - 4B \end{aligned}$$

This gives us the solution $A = -3/7$ and $B = 10/7$, so

$$\frac{x-7}{x^2-x-12} = \frac{-3/7}{x-4} + \frac{10/7}{x+3}$$

We could also find this by using the fact that

$$x - 7 = A(x+3) + B(x-4)$$

for all values of x. If we use $x = 4$, we get the equation $-3 = 7A$, so $A = -3/7$. With $x = -3$, $-10 = -7B$, so $B = 7/10$.

Note: With repeated factors, we need to allow for all powers of that factor. For example, if we had $(x-4)^2$, we would need to allow for 2 fractions, one with the denominator $x - 4$, the other with the denominator $(x-4)^2$

Examples

1. Find $\int \frac{x-7}{x^2-x-12}\,dx$.

   We found that

$$\begin{aligned} \int \frac{x-7}{x^2-x-12}\,dx &= \int \frac{-3/7}{x-4}\,dx + \int \frac{10/7}{x+3}\,dx \\ &= -\frac{3}{7}\ln|x-4| + \frac{10}{7}\ln|x+3| + C \end{aligned}$$

2. Find $\displaystyle\int \frac{4x^2-6x+2}{x^2(x-1)(x+3)}\,dx$.

We first need to carry out PFD. Let A, B, C, and D satisfy

$$\frac{4x^2-6x+2}{x^2(x-1)(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+3}$$

Multiplying by the denominator, we get:

$$4x^2-6x+2 = Ax(x-1)(x+3) + B(x-1)(x+3) + Cx^2(x+3) + Dx^2(x-1)$$

Using $x=0$ gives us the equation $2=-3B$, so $B=-2/3$. Using $x=1$ gives $0=4C$, so $C=0$. Letting $x=-3$ gives $56=-36D$, so $D=-56/36=-14/9$. To get A, we need to use another value of x in the equation

$$4x^2-6x+2 = Ax(x-1)(x+3) - \frac{2}{3}(x-1)(x+3) - \frac{14}{9}x^2(x-1)$$

Using $x=-1$ gives us the equation $12 = 4A + \frac{8}{3} + \frac{28}{9}$, so $A = \frac{14}{9}$.

Now, integrating, we get:

$$\begin{aligned}\int \frac{4x^2-6x+2}{x^2(x-1)(x+3)}\,dx &= \int \frac{14}{9}\frac{1}{x}\,dx - \int \frac{2}{3}x^{-2}\,dx - \int \frac{14}{9}\frac{1}{x+3}\,dx \\ &= \frac{14}{9}\ln|x| + \frac{2}{3}x^{-1} - \frac{14}{9}\ln|x+3| + C\end{aligned}$$

(Note: $C=0$ because (x-1) can be factored out of both the numerator and the denominator! This means that if we had reduced the fraction first, it might have been a little bit quicker.)

3. Find $\displaystyle\int \frac{33x^2-7x+70}{(3x-2)(x^2+4)}\,dx$

We want to rewrite

$$\frac{33x^2-7x+70}{(3x-2)(x^2+4)} = \frac{A}{3x-2} + \frac{Bx+C}{x^2+4}$$ Multiplying by the denominator on the LHS, we get

$$33x^2-7x+70 = A(x^2+4) + (Bx+C)(3x-2)$$

Letting $x=\frac{2}{3}$, we get that $A=18$. To get B and C, we need to compare coefficients. Starting with the coefficients for x^2, we get

$$\begin{aligned}33 &= A+3B = 18+3B \\ B &= 5\end{aligned}$$

To find C, we can compare the constants (you could compare the coefficients for x instead). Then,

$$\begin{aligned}70 &= 4A-2C = 72-2C \\ C &= 1\end{aligned}$$

This gives us that

$$\frac{33x^2-7x+70}{(3x-2)(x^2+4)} = \frac{18}{3x-2} + \frac{5x+1}{x^2+4} = \frac{18}{3x-2} + \frac{5x}{x^2+4} + \frac{1}{x^2+4}$$

Then, the integral is

$$\begin{aligned}\int \frac{33x^2-7x+70}{(3x-2)(x^2+4)}\,dx &= \int \frac{18}{3x-2}\,dx + \int \frac{5x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx \\ &= 6\ln\left|3x-2\right| + \frac{5}{2}\ln\left(x^2+4\right) + \frac{1}{2}\tan^{-1}\frac{x}{2} + C\end{aligned}$$

4. Find $\displaystyle\int \frac{\cos x}{\sin^4 x - 16}\,dx$

First, let $u = \sin x$. Then, $du = \cos x$, so we can rewrite the integral as

$$\int \frac{\cos x}{\sin^4 x - 16}\,dx = \int \frac{du}{u^4 - 16}$$

Therefore, we want to find A, B, C, and D such that:

$$\frac{1}{u^4 - 16} = \frac{A}{u+2} + \frac{B}{u-2} + \frac{Cx + D}{u^2 + 4}$$
$$1 = A(u-2)(u^2+4) + B(u+2)(u^2+4) + (Cx+D)(u+2)(u-2)$$

Letting $u = 2$, we get $1 = 32B$, so $B = \frac{1}{32}$.

Letting $u = -2$, we get that $1 = -32A$, so $A = -\frac{1}{32}$.

Letting $u = 0$, we get that $1 = -8A + 8B + -4D = \frac{1}{4} + \frac{1}{4} - 4D$, so $\frac{1}{2} = -4D$ and $D = -\frac{1}{8}$.

Finally, choosing $u = 1$ (You can actually use any other value of u that we haven't used yet. I chose 1 because it's simple to use.), we get $1 = -5A + 15B - 3(C + D) = \frac{5}{32} + \frac{15}{32} - 3C + \frac{3}{8}$, so $3C = \frac{5}{8} + \frac{3}{8} - 1 = 0$ and $C = 0$. Therefore, our integral becomes

$$\begin{aligned}\int \frac{\cos x}{\sin^4 x - 16}\,dx &= \int \frac{du}{u^4 - 16} \\ &= -\frac{1}{32}\int \frac{du}{u+2} + \frac{1}{32}\int \frac{du}{u-2} - \frac{1}{8}\int \frac{du}{u^2+4} \\ &= -\frac{1}{32}\ln\left|u+2\right| + \frac{1}{32}\ln\left|u-2\right| - \frac{1}{16}\tan^{-1}\left(\frac{u}{2}\right) + C \\ &= -\frac{1}{32}\ln\left|\sin x + 2\right| + \frac{1}{32}\ln\left|\sin x - 2\right| - \frac{1}{16}\tan^{-1}\left(\frac{\sin x}{2}\right) + C\end{aligned}$$

5. Find $\displaystyle\int \frac{x^6 - 7x^4 + 11x^3 - 13x^2 + x - 6}{x^3 - 2x^2}\,dx$

Since this is an improper rational function, we first need to carry out long division. Doing this, we get that

$$\frac{x^6 - 7x^4 + 11x^3 - 13x^2 + x - 6}{x^3 - 2x^2} = x^3 + 2x^2 - 3x + 5 + \frac{-3x^2 + x - 6}{x^2(x-2)}$$

We need to carry out partial fraction decomposition for $\frac{-3x^2+x-6}{x^2(x-2)}$, so we need to find A, B, and C such that

$$\frac{-3x^2 + x - 6}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$
$$-3x^2 + x - 6 = Ax(x-2) + B(x-2) + Cx^2$$

Using $x = 0$ gives us that $-6 = -2B$, so $B = 3$. $x = 2$ gives us that $-12 + 2 - 6 = -16 = 4C$, so $C = -4$. To find A, we need to choose another value of x. Let's choose $x = 1$, so we get the equation $-3 + 1 - 6 = -8 = -A - B + C = -A - 3 - 4 = -A - 7$, so $A = 1$. As a result, our integral becomes:

$$\begin{aligned}\int \frac{x^6 - 7x^4 + 11x^3 - 13x^2 + x - 6}{x^3 - 2x^2}\,dx &= \int \left(x^3 + 2x^2 - 3x + 5\right)dx + \int \frac{dx}{x} \\ &\quad + \int 3x^{-2}\,dx - \int \frac{4}{x-2}\,dx \\ &= \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 5x + \ln\left|x\right| \\ &\quad - 3x^{-1} - 4\ln\left|x-2\right| + C\end{aligned}$$