# Section 7.4, Rationalizing Substitutions

Homework: 7.4 #1–29 odds

Radicals often cause problems when integrating, so we will cover which substitution to use in various circumstances.

## 1 Integrals involving $\sqrt[n]{ax+b}$

In circumstances when $\sqrt[n]{ax+b}$ appears, let $u = \sqrt[n]{ax+b}$.

**Examples**

1. Find $\int \frac{x^2+3x}{\sqrt{x+4}}\,dx$
   Let $u = \sqrt{x+4}$. Then, $du = \frac{1}{2}(x+4)^{-1/2}\,dx$ and $x = u^2 - 4$. Therefore,

$$\begin{aligned}\int \frac{x^2+3x}{\sqrt{x+4}}\,dx &= \int 2\big((u^2-4)^2 + (u^2-4)\big)\,du \\ &= \int \big(2u^4 - 10u^2 + 8\big)\,du \\ &= \frac{2}{5}u^5 - \frac{10}{3}u^3 + 8u + C \\ &= \frac{2}{5}(x+4)^{5/2} - \frac{10}{3}(x+4)^{3/2} + 8(x+4)^{1/2} + C\end{aligned}$$

2. Find $\int \frac{\sqrt{x}}{x+1}\,dx$.
   Let $u = \sqrt{x}$. Then, $du = \frac{1}{2}x^{-1/2}\,dx$ and $x = u^2$.

$$\begin{aligned}\int \frac{\sqrt{x}}{x+1}\,dx &= \int \frac{2u^2}{u^2+1}\,du \\ &= \int \left(2 - \frac{2}{u^2+1}\right)du \\ &= 2u - 2\arctan u + C \\ &= 2\sqrt{x} - 2\arctan\sqrt{x} + C\end{aligned}$$

## 2 Integrals involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$

All three of these forms will involve trigonometric substitutions:

1. For $\sqrt{a^2-x^2}$, let $x = a\sin t$ for $-\pi/2 \le t \le \pi/2$.
2. For $\sqrt{a^2+x^2}$, let $x = a\tan t$ for $-\pi/2 < t < \pi/2$.
3. For $\sqrt{x^2-a^2}$, let $x = a\sec t$ for $0 \le t \le \pi$, $t \ne \pi/2$.

These substitutions result in:

1. $\sqrt{a^2-x^2} = a\cos t$.
2. $\sqrt{a^2+x^2} = a\sec t$.

3. $\sqrt{x^2 - a^2} = \pm a \tan t$.

The formulas from Section 6.8 may be helpful, especially for simplification of the expression.

**Examples**

Find $\int \frac{x^2}{\sqrt{16-x^2}}\, dx$

Let $x = 4\sin t$. Then, $\sqrt{16-x^2} = 4\cos t$, $dx = 4\cos t\, dt$ and $t = \sin^{-1}(x/4)$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}} &= \int \frac{16\sin^2 t}{4\cos t} 4\cos t\, dt \\
&= \int 16\sin^2 t\, dt \\
&= \int 8(1-\cos 2t)\, dt, \quad \text{where we used } sin^2 t = (1-\cos 2x)/2 \\
&= 8t - 4\sin 2t + C \\
&= 8\sin^{-1}\frac{x}{4} - 4\sin\left(2\sin^{-1}\frac{x}{4}\right) + C \\
&= 8\sin^{-1}\frac{x}{4} - 8\sin\left(\sin^{-1}\frac{x}{4}\right)\cos\left(\sin^{-1}\frac{x}{4}\right) + C \\
&= 8\sin^{-1}\frac{x}{4} - 8\cdot\frac{x}{4}\sqrt{1-(x/4)^2} + C \\
&= 8\sin^{-1}\frac{x}{4} - \frac{x}{2}\sqrt{16-x^2} + C
\end{aligned}$$

# 3 Completing the Square

Sometimes, we are given radicals that don't look quite like what we need for the substitutions that we have talked about so far in this section. At times, we can still find a $u$-substitution that will transform the radical into what we need. This method will work when there is a quadratic expression under the radical.

**Example**

Find $\int \frac{3x}{\sqrt{x^2+4x-5}}\, dx$.

Note that $x^2 + 4x - 5 = x^2 + 4x + 4 - 9 = (x+2)^2 - 9$. Then, letting $x + 2 = 3\sec t$ (so $dx = 3\sec t\tan t\, dt$, $x = 3\sec t - 2$, $\sqrt{x^2+4x-5} = \pm 3\tan t$, and $t = \sec^{-1}\frac{x+2}{3}$.)

$$\begin{aligned}
\int \frac{3x}{\sqrt{x^2+4x-5}}\, dx &= \int \frac{3(3\sec t - 2)}{3\tan t} 3\sec t\tan t\, dt \\
&= 9\int \sec^2 t\, dt - 6\int \sec t\, dt \\
&= 9\tan t - 6\ln|\sec t + \tan t| + C \\
&= 3\sqrt{x^2+4x-5} - 6\ln\left|\frac{x+2}{3} + \frac{\sqrt{x^2+4x-5}}{3}\right| + C \\
&= 3\sqrt{x^2+4x-5} - 6\ln\left|x+2+\sqrt{x^2+4x-5}\right| + K,
\end{aligned}$$

where $K = C + 6\ln 3$.