Section 7.3, Some Trigonometric Integrals

Homework: 7.3 #1–31 odds

We will look at five commonly encountered types of trigonometric integrals:

1. $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$
2. $\int \sin^m x \cos^n x\, dx$
3. $\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$, and $\int \cos mx \cos nx\, dx$
4. $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$
5. $\int \tan^m x \sec^n\, dx$ and $\int \cot^m x \csc^n x\, dx$

We will demonstrate how to calculate these by example. Throughout this section, we will be using many trigonometric identities, including:

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\sin mx \cos nx = \frac{1}{2}\big[\sin(m+n)x + \sin(m-n)x\big]$$
$$\sin mx \sin nx = -\frac{1}{2}\big[\cos(m+n)x - \cos(m-n)x\big]$$
$$\cos mx \cos nx = \frac{1}{2}\big[\cos(m+n)x + \cos(m-n)x\big]$$

1 Integrals of the form $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$

We will look at examples when n is odd and when n is even. When n is odd, we will use $\sin^2 x + \cos^2 x = 1$. When n is even, we will use either $\sin^2 x = \frac{1-\cos 2x}{2}$ or $\cos^2 x = \frac{1+\cos 2x}{2}$.

Examples

1. Find $\int \cos^5 x\, dx$.
   We will use the identity $\cos^2 x = 1 - \sin^2 x$, so we will substitute $\cos^4 x = (1 - \sin^2 x)^2$.

$$\begin{aligned}\int \cos^5 x\, dx &= \int (1-\sin^2 x)^2 \cos x\, dx \\ &= \int \big(1 - 2\sin^2 x + \sin^4 x\big) \cos x\, dx \\ &= \int \big(\cos x - 2\sin^2 x \cos x + \sin^4 x \cos x\big)\, dx \\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C\end{aligned}$$

2. Find $\int \sin^4 x\, dx$
We will start by using $\sin^2 x = \frac{1-\cos 2x}{2}$.

$$\begin{aligned}
\int \sin^4 x\, dx &= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx \\
&= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{4}\int \cos^2 2x\, dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}\, dx \\
&= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C \\
&= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C,
\end{aligned}$$

where we also used that $\cos^2 x = \frac{1+\cos 2x}{2}$ in the third-to-last line.

2 Integrals of the form $\int \sin^m x \cos^n x\, dx$

If either m or n is an odd positive integer, we will use the identity $\sin^2 x + \cos^2 x = 1$. If both m and n are even and positive, we will use the half-angle identities.

Examples

1. Find $\int \cos^5 x \sin^{-4} x\, dx$
Since the exponent for cosine is odd, we will replace $\cos^4 x$ by $(1-\sin^2 x)^2 = 1-2\sin^2 x+\sin^4 x$:

$$\begin{aligned}
\int \cos^5 x \sin^{-4} x\, dx &= \int \cos x(1 - 2\sin^2 x + \sin^4 x)\sin^{-4} x\, dx \\
&= \int \cos x \sin^{-4} x\, dx - 2\int \cos x \sin^{-2} x\, dx + \int \cos x\, dx \\
&= -\frac{1}{3}(\sin x)^{-3} + 2(\sin x)^{-1} + \sin x + C
\end{aligned}$$

(Be careful with notation, since $\sin^{-1} x$ refers to the inverse sine function, not $1/(\sin x)$.)

2. Find $\int \cos^2 x \sin^4 x\, dx$.

We will substitute $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2$. Then,

$$\begin{aligned}
\int \cos^2 x \sin^4 x\, dx &= \int \frac{1+\cos 2x}{2}\left(\frac{1-\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1+\cos 2x)(1 - 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{8}\int (1 - \cos 2x - \cos^2 2x + \cos^3 2x)\, dx \\
&= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 2x\, dx - \frac{1}{8}\int \cos^2 2x\, dx + \frac{1}{8}\int \cos^3 2x\, dx \\
&= \frac{x}{8} - \frac{1}{16}\sin 2x - \frac{1}{8}\int \frac{1+\cos 4x}{2}\, dx + \frac{1}{8}\int \cos 2x(1-\sin^2 2x)\, dx \\
&= \frac{x}{8} - \frac{1}{16}\sin 2x - \frac{x}{16} - \frac{1}{64}\sin 4x + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C \\
&= \frac{x}{16} - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C
\end{aligned}$$

3 Integrals of the form $\int \sin mx \cos nx\, dx$, $\int \sin mx \sin nx\, dx$, and $\int \cos mx \cos nx\, dx$

Here, we will use the identities $\sin mx \cos nx = \frac{1}{2}\big[\sin(m+n)x + \sin(m-n)x\big]$, $\sin mx \sin nx = -\frac{1}{2}\big[\cos(m+n)x - \cos(m-n)x\big]$, and $\cos mx \cos nx = \frac{1}{2}\big[\cos(m+n)x + \cos(m-n)x\big]$.

Examples

1. Find $\int \sin 4x \cos 5x\, dx$
   Since $\sin mx \cos nx = \frac{1}{2}\big[\sin(m+n)x + \sin(m-n)x\big]$, we will use this with $m = 4$ and $n = 5$:

$$\begin{aligned}\int \sin 4x \cos 5x\, dx &= \frac{1}{2}\int \big(\sin 9x + \sin(-x)\big)\, dx \\ &= -\frac{1}{18}\cos 9x + \frac{1}{2}\cos x + C,\end{aligned}$$

   where we used that $\cos(-x) = \cos x$.

2. Find $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx$, where m and n are positive integers.
   First, consider $m \neq n$. Then,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx\, dx &= -\frac{1}{2}\int_{-\pi}^{\pi} \big[\cos(m+n)x - \cos(m-n)x\big]\, dx \\ &= -\frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x - \frac{1}{m-n}\sin(m-n)\right]\Bigg|_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

   If $m = n$, then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx\, dx &= -\frac{1}{2}\int_{-\pi}^{\pi} \big[\cos(2m)x - 1\big]\, dx \\ &= -\frac{1}{2}\left[\frac{1}{2m}\sin(2m)x - x\right]\Bigg|_{-\pi}^{\pi} \\ &= \pi\end{aligned}$$

4 Integrals of the form $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$

In the tangent case, we will use $\tan^2 x = \sec^2 x - 1$. In the cotangent case, we will use $\cot^2 x = \csc^2 x - 1$. Here, we will only replace $\tan^2 x$ or $\cot^2 x$, distribute, integrate what we can, then repeat as necessary.

Examples

1. Find $\int \tan^4 x\, dx$

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x(\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\ &= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\, dx \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C\end{aligned}$$

2. Find $\int \cot^5 x\,dx$

$$\begin{aligned}\int \cot^5 x\,dx &= \int \cot^3 x(\csc^2 x - 1)\,dx \\ &= \int \cot^3 x \csc^2 x - \int \cot^3 x\,dx \\ &= -\frac{1}{4}\cot^4 x - \int \cot x(\csc^2 x - 1)\,dx \\ &= -\frac{1}{4}\cot^4 x - \int \cot x \csc^2 x\,dx + \int \frac{\cos x}{\sin x}\,dx \\ &= -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \ln|\sin x| + C\end{aligned}$$

5 Integrals of the form $\int \tan^m x \sec^n\,dx$ and $\int \cot^m x \csc^n x\,dx$

If n is even, use either $\sec^2 x = \tan^2 x + 1$ or $\csc^2 x = \cot^2 x + 1$ to replace all but 2 powers of $\sec x$ or $\csc x$, then you can use a u-substitution to integrate.

If m is odd, we will use that the derivative of $\sec x$ is $\sec x \tan x$ (or the derivative of $\csc x$ is $-\csc x \cot x$), and replace $m-1$ powers of either tangent or cotangent using a Pythagorean identity.

Examples

1. Find $\int \tan^{1/2} x \sec^4 x\,dx$

$$\begin{aligned}\int \tan^{1/2} x \sec^4 x\,dx &= \int \tan^{1/2} x(\tan^2 + 1)\sec^2 x\,dx \\ &= \int \tan^{5/2} x \sec^2 x\,dx + \int \tan^{1/2} x \sec^2 x\,dx \\ &= \frac{2}{7}\tan^{7/2} x + \frac{2}{3}\tan^{3/2} x + C\end{aligned}$$

2. Find $\int \cot^3 x \csc^{3/2} x\,dx$

$$\begin{aligned}\int \cot^3 x \csc^{3/2} x\,dx &= \int \cot x(\csc^2 x - 1)\csc^{3/2} x\,dx \\ &= \int \cot x \csc x \csc^{5/2} x\,dx - \int \cot x \csc x \csc^{1/2} x\,dx \\ &= -\frac{2}{7}\csc^{7/2} x + \frac{2}{3}\csc^{3/2} + C\end{aligned}$$