Section 7.2, Integration by Parts

Homework: 7.2 #1–57 odd

So far, we have been able to integrate some products, such as $\int xe^{x^2}\,dx$. They have been able to be solved by a u-substitution. However, what happens if we can't solve it by a u-substitution? For example, consider $\int xe^x\,dx$. For this, we will need a technique called **integration by parts**.

From the product rule for derivatives, we know that $D_x[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$. Rearranging this and integrating, we see that:

$$u(x)v'(x) = D_x[u(x)v(x)] - u'(x)v(x)$$
$$\int u(x)v'(x)\,dx = \int D_x[u(x)v(x)]\,dx - \int u'(x)v(x)\,dx$$

This gives us the formula needed for integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx, \quad \text{or, we can write it as}$$
$$\int u\,dv = uv - \int v\,du$$

In this section, we will practice choosing u and v properly.

Examples
Perform each of the following integrations:

1. $\int xe^x\,dx$
   Let $u = x$, and $dv = e^x\,dx$. Then, $du = dx$ and $v = e^x$. Using our formula, we get that
   $$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

2. $\int \frac{\ln x}{\sqrt{x}}\,dx$
   Since we do not know how to integrate $\ln x$, let $u = \ln x$ and $dv = x^{-1/2}$. Then $du = 1/x$ and $v = 2x^{1/2}$, so
   $$\begin{aligned}\int \frac{\ln x}{\sqrt{x}}\,dx &= 2x^{1/2}\ln x - \int 2x^{-1/2}\,dx \\ &= 2x^{1/2}\ln x - 4x^{1/2} + C\end{aligned}$$

3. $\int \arctan x\,dx$
   We aren't given a (obvious) product here, but we don't have an integration formula for $\arctan x$. So, we can think of the integral as $\int 1\cdot\arctan x\,dx$, and let $u = \arctan x$ and $dv = 1\,dx$. Then, $du = \frac{dx}{1+x^2}$ and $v = x$. Using the formula,
   $$\begin{aligned}\int \arctan x\,dx &= x\arctan x - \int \frac{x}{1+x^2}\,dx \\ &= x\arctan x - \frac{1}{2}\ln(1+x^2) + C\end{aligned}$$
   A similar technique can be used to integrate other inverse trigonometric functions, as well as the natural logarithm function.

4. $\int x^2 \sin(2x)\,dx$
Let $u = x^2$ and $dv = \sin(2x)\,dx$. Then, $du = 2x\,dx$ and $v = -\frac{\cos(2x)}{2}$. Using our formula,

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \int x\cos(2x)\,dx$$

Since we do not know the integral of $x\cos(2x)$, we will repeat integration by parts with $u = x$ and $dv = \cos(2x)\,dx$. Then, $du = dx$ and $v = \frac{\sin(2x)}{2}$. We then get

$$\begin{aligned}\int x\cos(2x)\,dx &= \frac{x}{2}\sin(2x) - \int \frac{\sin(2x)}{2}\,dx \\ &= \frac{x}{2}\sin(2x) + \frac{\cos(2x)}{4} + C\end{aligned}$$

As a result, our final answer is

$$\int x^2 \sin(2x)\,dx = -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{\cos(2x)}{4} + C$$

5. $\int e^x \cos x\,dx$
Let $u = e^x$ and $dv = \cos x\,dx$. Then, $du = e^x\,dx$ and $v = \sin x$, so

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

We will repeat integration by parts with $u = e^x$ and $dv = \sin x\,dx$, so $du = e^x\,dx$ and $v = -\cos x$. From our formula, we see:

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \int e^x \sin x\,dx \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \quad \text{Solving for the integral, we get} \\ 2\int e^x \cos x\,dx &= e^x \sin x + e^x \cos x \\ \int e^x \cos x\,dx &= \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C\end{aligned}$$

6. Derive a reduction formula for $\int \cos^n x\,dx$ when $n \geq 2$.
Let $u = \cos^{n-1} x$ and $dv = \cos x\,dx$. Then, $du = -(n-1)\cos^{n-2} x \sin x\,dx$ and $v = \sin x$. By our formula,

$$\begin{aligned}\int \cos^n x\,dx &= \sin x \cos^{n-1} x + \int (n-1)\cos^{n-2} x \sin^2 x\,dx \\ &= \sin x \cos^{n-1} x + \int (n-1)\cos^{n-2} x(1-\cos^2 x)\,dx \\ &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx. \\ &\quad \text{Solving for original integral, we get that} \\ n\int \cos^n x\,dx &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx \\ \int \cos^n x\,dx &= \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx\end{aligned}$$