Section 6.9, The Hyperbolic Functions and Their Inverses

Homework: 6.9 #1-51 odds

In this section, we will define the six **hyperbolic functions**, which are combinations of e^x and e^{-x}.

1 Hyperbolic Functions

Hyperbolic sine, hyperbolic cosine, hyperbolic tangent, and their reciprocals are:

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\text{csch } x &= \frac{1}{\sinh x} \\
\text{sech } x &= \frac{1}{\cosh x} \\
\coth x &= \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}
\end{aligned}$$

sinh and cosh satisfy the identity

$$\cosh^2 x - \sinh^2 x = 1.$$

We can see this by writing it out:

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{2} - \frac{e^{2x} - 2 + e^{-2x}}{2} = 1.$$

Note that sinh is an odd function since $\sinh(-x) = -\sinh x$ and cosh is an even function since $\cosh(-x) = \cosh x$.

The graphs of four of these functions are shown in Figure 3 on page 375 of the book (also sketched on the board in class).

Example

Verify that $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$. (#10)

Note that:

$$\begin{aligned}
\frac{\tanh x - \tanh y}{1 - \tanh x \tanh y} &= \frac{\frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{e^y - e^{-y}}{e^y + e^{-y}}}{1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^y - e^{-y}}{e^y + e^{-y}}} \cdot \frac{(e^x + e^{-x})(e^y + e^{-y})}{(e^x + e^{-x})(e^y + e^{-y})} \\
&= \frac{(e^x - e^{-x})(e^y + e^{-y}) - (e^y - e^{-y})(e^x + e^{-x})}{(e^x + e^{-x})(e^y + e^{-y}) - (e^x - e^{-x})(e^y - e^{-y})} \\
&= \frac{e^{x+y} + e^{x-y} - e^{y-x} - e^{-x-y} - e^{x+y} - e^{y-x} + e^{x-y} + e^{-x-y}}{e^{x+y} + e^{x-y} + e^{y-x} + e^{-x-y} - e^{x+y} + e^{x-y} + e^{y-x} - e^{-x-y}} \\
&= \frac{2e^{x-y} - 2e^{y-x}}{2e^{x-y} + 2e^{y-x}} \\
&= \frac{e^{x-y} - e^{y-x}}{e^{x-y} - e^{y-x}} \\
&= \tanh(x - y)
\end{aligned}$$

(To show identities, it is normally easier to start with the more complicated side and simplify it.)

2 Derivatives

The derivatives of sinh and cosh can be computed as:

$$D_x \sinh x = D_x \left[\frac{e^x - e^{-x}}{2} \right] = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$D_x \cosh x = D_x \left[\frac{e^x + e^{-x}}{2} \right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

The other derivatives can be calculated using the quotient rule:

$$\begin{aligned}
D_x \sinh x &= \cosh x \\
D_x \cosh x &= \sinh x \\
D_x \tanh x &= \text{sech }^2 x \\
D_x \text{csch } x &= -\text{csch } x \coth x \\
D_x \text{sech } x &= -\text{sech } \tanh x \\
D_x \coth x &= -\text{csch }^2 x
\end{aligned}$$

Note that these are similar to the derivatives of trigonometric functions (with the exception of a few negative signs).

Examples

1. Calculate $D_x[\cosh^3(\sin x)]$.

$$D_x[\cosh^3(\sin x)] = 3\cosh^2(\sin x) \cdot \sinh(\sin x) \cdot \cos x$$

2. Calculate $\int \cosh(3x+1)\,dx$.

$$\int \cosh(3x+1)\,dx = \frac{1}{3}\sinh(3x+1) + C$$

3. Calculate $\int \tanh x\,dx$.

$$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \ln|\cosh x| + C = \ln\big(\cosh x\big) + C$$

3 Inverse Hyperbolic Functions

All of the hyperbolic functions have inverses for an appropriate domain (for cosh and sech , we restrict the domain to $x \geq 0$. The rest hold for all real numbers.). The four we will use most often are:

$$\begin{aligned}
\sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right) \\
\cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right) \quad x \geq 1 \\
\tanh^{-1} x &= \frac{1}{2}\ln\frac{1+x}{1-x}, \quad -1 < x < 1 \\
\text{sech }^{-1} x &= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1
\end{aligned}$$

Proof of the $\sinh^{-1}$ formula: Using the procedure for finding inverse functions, set $y = \frac{e^x - e^{-x}}{2}$. Solving for x, we get:

$$\begin{aligned} 2y &= e^x - e^{-x} \\ 0 &= e^x - 2y - e^{-x} \\ 0 &= e^{-x}\left(e^{2x} - 2ye^x - 1\right) = e^{-x}\left(\left(e^x\right)^2 - 2ye^x - 1\right). \end{aligned}$$

e^{-x} never equals zero, but we can use the quadratic formula to solve for e^x in the second factor.

$$e^x = \frac{2y \pm \sqrt{(2y)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{2y \pm \sqrt{4y^2 + 4}}{2}.$$

Since e^x cannot be negative, we can ignore the "$-$" answer.

$= y + \sqrt{y^2 + 1}$. Solving for x, we get:

$$x = \ln\left(y + \sqrt{y^2 + 1}\right)$$

So, $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

The book shows the proof of the formula for $\cosh^{-1}$.

We can use the formulas to get the derivatives for the inverse hyperbolic functions:

$$\begin{aligned} D_x \sinh^{-1} x &= \frac{1}{\sqrt{x^2 + 1}} \\ D_x \cosh^{-1} x &= \frac{1}{\sqrt{x^2 - 1}} \quad x > 1 \\ D_x \tanh^{-1} x &= \frac{1}{1 - x^2}, \quad -1 < x < 1 \\ D_x \text{sech}^{-1} x &= \frac{-1}{x\sqrt{1 - x^2}}, \quad 0 < x < 1 \end{aligned}$$

Proof of the formula for $D_x \sinh^{-1} x$:

$$\begin{aligned} D_x \sinh^{-1} x &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{2x}{2\sqrt{x^2 + 1}}\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

Example
Calculate y' if $y = x^3 \sinh^{-1}(x^6)$.

$$y' = x^3 \cdot \frac{6x^5}{\sqrt{x^{12} + 1}} + 3x^2 \sinh^{-1}(x^6) = \frac{6x^8}{\sqrt{x^{12} + 1}} + 3x^2 \sinh^{-1}(x^6)$$