In this section, we will define the six hyperbolic functions, which are combinations of \( e^x \) and \( e^{-x} \).

### 1 Hyperbolic Functions

Hyperbolic sine, hyperbolic cosine, hyperbolic tangent, and their reciprocals are:

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\text{csch} x &= \frac{1}{\sinh x} \\
\text{sech} x &= \frac{1}{\cosh x} \\
\coth x &= \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}
\end{align*}
\]

sinh and cosh satisfy the identity

\[
\cosh^2 x - \sinh^2 x = 1.
\]

We can see this by writing it out:

\[
\begin{align*}
\cosh^2 x - \sinh^2 x &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1.
\end{align*}
\]

Note that sinh is an odd function since \( \sinh(-x) = -\sinh x \) and cosh is an even function since \( \cosh(-x) = \cosh x \).

The graphs of four of these functions are shown in Figure 3 on page 375 of the book (also sketched on the board in class).

**Example**

Verify that \( \tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y} \). (#10)

Note that:

\[
\begin{align*}
\tanh x - \tanh y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} - \frac{e^y - e^{-y}}{e^y + e^{-y}} \\
&= \frac{(e^x - e^{-x})(e^y + e^{-y}) - (e^y - e^{-y})(e^x + e^{-x})}{(e^x + e^{-x})(e^y + e^{-y})} \\
&= \frac{e^x + e^{-x} + e^{x+y} + e^{y-x} - e^{-x+y} - e^{y-x} - e^{x+y} + e^x - e^{-y}}{(e^x + e^{-x})(e^y + e^{-y})} \\
&= \frac{2e^{x-y}}{2e^{x+y} - 2e^{y-x}} \\
&= \frac{e^{x-y}}{e^{x+y} - e^{y-x}} \\
&= \tanh(x - y)
\end{align*}
\]

(To show identities, it is normally easier to start with the more complicated side and simplify it.)
2 Derivatives

The derivatives of sinh and cosh can be computed as:

\[
D_x \sinh x = D_x \left[ \frac{e^x - e^{-x}}{2} \right] = \frac{e^x + e^{-x}}{2} = \cosh x
\]

\[
D_x \cosh x = D_x \left[ \frac{e^x + e^{-x}}{2} \right] = \frac{e^x - e^{-x}}{2} = \sinh x
\]

The other derivatives can be calculated using the quotient rule:

\[
D_x \sinh x = \cosh x \\
D_x \cosh x = \sinh x \\
D_x \tanh x = \text{sech}^2 x \\
D_x \text{csch} x = -\csc h x \coth x \\
D_x \text{sech} x = -\text{sech} \tanh x \\
D_x \coth x = -\csc h^2 x
\]

Note that these are similar to the derivatives of trigonometric functions (with the exception of a few negative signs).

Examples

1. Calculate \( D_x [\cosh^3 (\sin x)] \).

\[
D_x [\cosh^3 (\sin x)] = 3 \cosh^2 (\sin x) \cdot \sinh (\sin x) \cdot \cos x
\]

2. Calculate \( \int \cosh (3x + 1) \, dx \).

\[
\int \cosh (3x + 1) \, dx = \frac{1}{3} \sinh (3x + 1) + C
\]

3. Calculate \( \int \tanh x \, dx \).

\[
\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx = \ln |\cosh x| + C = \ln (\cosh x) + C
\]

3 Inverse Hyperbolic Functions

All of the hyperbolic functions have inverses for an appropriate domain (for \( \cosh \) and sech, we restrict the domain to \( x \geq 0 \). The rest hold for all real numbers.). The four we will use most often are:

\[
\sinh^{-1} x = \ln (x + \sqrt{x^2 + 1}) \\
\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}) \quad x \geq 1 \\
\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}, \quad -1 < x < 1 \\
\text{sech}^{-1} x = \ln \left( \frac{1 + \sqrt{1 - x^2}}{x} \right), \quad 0 < x \leq 1
\]
**Proof of the sinh\(^{-1}\) formula:** Using the procedure for finding inverse functions, set \(y = \frac{e^x - e^{-x}}{2}\). Solving for \(x\), we get:

\[
2y = e^x - e^{-x} \\
0 = e^x - 2y - e^{-x} \\
0 = e^{-x}(e^{2x} - 2ye^x - 1) = e^{-x}(e^x)^2 - 2ye^x - 1).
\]

\(e^{-x}\) never equals zero, but we can use the quadratic formula to solve for \(e^x\) in the second factor.

\[
e^x = \frac{2y \pm \sqrt{(2y)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{2y \pm \sqrt{4y^2 + 4}}{2}.
\]

Since \(e^x\) cannot be negative, we can ignore the “−” answer.

\[
x = \ln \left( y + \sqrt{y^2 + 1} \right)
\]

So, \(\sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right)\).

The book shows the proof of the formula for \(\cosh^{-1}\).

We can use the formulas to get the derivatives for the inverse hyperbolic functions:

\[
D_x \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}} \\
D_x \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \quad x > 1 \\
D_x \tanh^{-1} x = \frac{1}{1 - x^2}, \quad -1 < x < 1 \\
D_x \text{sech}^{-1} x = \frac{-1}{x\sqrt{1 - x^2}}, \quad 0 < x < 1
\]

**Proof of the formula for \(D_x \sinh^{-1} x\):**

\[
D_x \sinh^{-1} x = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left( 1 + \frac{2x}{2\sqrt{x^2 + 1}} \right) \\
= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \\
= \frac{1}{\sqrt{x^2 + 1}}
\]

**Example**

Calculate \(y'\) if \(y = x^3 \sinh^{-1}(x^6)\).

\[
y' = x^3 \cdot \frac{6x^5}{\sqrt{x^{12} + 1}} + 3x^2 \sinh^{-1}(x^6) = \frac{6x^8}{\sqrt{x^{12} + 1}} + 3x^2 \sinh^{-1}(x^6)
\]