Section 6.8, The Inverse Trigonometric Functions and Their Derivatives

Homework: 6.8 #1-73 odds

Since all trigonometric functions are 2π-periodic, they are not one-to-one. Therefore, in order to define inverse functions, we need to restrict the domain.

For sine and cosine, we restrict their domains to $[-\pi/2, \pi/2]$ and $[0, \pi]$, respectively. Then,

$$x = \sin^{-1} y \Leftrightarrow y = \sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$x = \cos^{-1} y \Leftrightarrow y = \cos x, \quad 0 \leq x \leq \pi$$

For tangent and secant, the domains are restricted to $(-\pi/2, \pi/2)$ and $[0, \pi/2) \cup (\pi/2, \pi]$, respectively. Furthermore,

$$x = \tan^{-1} y \Leftrightarrow y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$x = \sec^{-1} y \Leftrightarrow y = \sec x, \quad 0 \leq x \leq \pi, x \neq \frac{\pi}{2}$$

This book has no need to define $\csc^{-1}$ or $\cot^{-1}$. Also, the domain of secant is sometimes restricted in different ways.

There is alternate notation for inverse trigonometric functions: $\arcsin x = \sin^{-1} x$, $\arccos x = \cos^{-1} x$, and $\arctan x = \tan^{-1} x$.

Examples
Calculate each of the following:

1. $\sin^{-1} \frac{1}{2}$
   Since $\sin(\pi/6) = 1/2$, $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$.
2. $\arccos(-\frac{\sqrt{2}}{2})$
   Since $\cos(3\pi/4) = -\sqrt{2}/2$, $\arccos(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$.
3. $\arccos(\cos(\frac{5\pi}{4}))$
   Since $5\pi/4$ is not in the range for arccos, we can use that $\cos(\frac{5\pi}{4}) = \cos(\frac{3\pi}{4})$. Therefore, $\arccos(\cos(\frac{5\pi}{4})) = \frac{3\pi}{4}$.
4. $\tan(\tan^{-1}(1))$
   $\tan(\tan^{-1}(1)) = 1$ since tan and $\tan^{-1}$ are inverse functions.
5. $\cos(\sec^{-1}(2))$
   We can use that $\sec x = 1/\cos x$, so $\cos(\sec^{-1}(2)) = \cos(\cos^{-1}(1/2)) = 1/2$.
6. $\cos\left(\sin^{-1} \frac{2}{3}\right)$
   We want the value of the cosine for the angle whose sine is 2/3. Drawing a right triangle with hypotenuse 1, we get that the two legs have lengths 2/3 and $\sqrt{1-(2/3)^2} = \sqrt{5}/3$, so $\cos\left(\sin^{-1} \frac{2}{3}\right) = \frac{\sqrt{5}}{3}$.

There are many identities that can be used for inverse trigonometric functions. Some of them are:
Theorem A

1. $\sin(\cos^{-1} x) = \sqrt{1-x^2}$
2. $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

3. $\sec(\tan^{-1} x) = \sqrt{1+x^2}$
4. $\tan(\sec^{-1} x) = \begin{cases} \sqrt{x^2-1} & x \geq 1 \\ -\sqrt{x^2-1} & x \leq -1 \end{cases}$

Sketch of Proof: The first two statements can be shown by drawing a triangle and using that $\sin^2\theta + \cos^2\theta = 1$. The last two use that $\sec^2\theta = 1 + \tan^2\theta$.

1 Derivatives

Theorem B

1. $D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$
2. $D_x \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$
3. $D_x \tan^{-1} x = \frac{1}{1+x^2}$
4. $D_x \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

Proof of (2): Let $y = \cos^{-1} x$. Then $x = cosy$ and $1 = -\sin y D_x y = -\sin(\cos^{-1} x) D_x y$. Solving for $D_x y$, we get:

$$D_x y = -\frac{1}{\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}$$

Proofs of the other statements are similar (the proof of the first one is the book).

Examples
Find the derivatives of each of the following functions:

1. $y = e^x \arccos(3x^3)$

$$y' = e^x \arccos(3x^3) + e^x \frac{9x^2}{\sqrt{1-9x^6}}$$

2. $f(x) = -\ln(\csc x + \cot x)$

$$\begin{aligned} f'(x) &= \frac{\csc x \cot x + \csc^2 x}{\csc x + \cot x} \\ &= \csc x \frac{\cot x + \csc x}{\csc x + \cot x} = \csc x \end{aligned}$$

3. $g(t) = (\sin^{-1}(2t))^2$

$$\begin{aligned} g'(t) &= 2\sin^{-1}(2t) \cdot \frac{2}{\sqrt{1-4t^2}} \\ &= \frac{4\sin^{-1}(2t)}{\sqrt{1-4t^2}} \end{aligned}$$

2 Integrals

The derivative formulas can be converted to integral formulas and generalized to

1. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

2. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

Proof of (1): Assume $a > 0$

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int \frac{1}{a\sqrt{1-(\frac{x}{a})^2}}\,dx. \text{ Let } u = x/a.\\ &= \int \frac{1}{\sqrt{1-u^2}}\,du\\ &= \sin^{-1}(u) + C = \sin^{-1}\left(\frac{x}{a}\right) + C\end{aligned}$$

Examples

1. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$ (#62)

$$\begin{aligned}\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx &= \sec^{-1}(x)\Big|_{\sqrt{2}}^{2}\\ &= \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}\end{aligned}$$

2. $\displaystyle\int \frac{e^{2t}}{9+4e^{4t}}\,dt$

Let $u = 2e^{2t}$. Then,

$$\begin{aligned}\int \frac{e^{2t}}{9+4e^{4t}}\,dt &= \frac{1}{4}\int \frac{1}{3^2+u^2}\,du\\ &= \frac{1}{4}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C\\ &= \frac{1}{12}\tan^{-1}\left(\frac{2e^{2t}}{3}\right) + C\end{aligned}$$

Note: This section also reviews some of the derivatives and integrals for trigonometric functions.