Section 6.6, First-Order Linear Differential Equations

Homework: 6.6 #1-21 odds

Consider the equation

$$\frac{dy}{dx} = 4x - \frac{6y}{x}.$$

This equation is not **separable**, since we cannot separate the variables to have dy multiplying all the terms with y on one side, and dx multiplying all the terms with x on the other. This kind of equation is called a **first-order linear differential equation**.

Our goal for this section will be to find a general solution for equations of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and find a particular solution if an initial condition is given.

Our first step in solving differential equations of this form is to multiply both sides by an integrating factor, $\exp(\int P(x)\,dx)$. Then,

$$\begin{aligned} e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}P(x)y &= e^{\int P(x)\,dx}Q(x). \text{ Note that the LHS is the derivative of } y\cdot e^{\int P(x)\,dx} \\ \frac{d}{dx}\left(y\cdot e^{\int P(x)\,dx}\right) &= e^{\int P(x)\,dx}Q(x). \text{ Then we integrate both sides:} \\ y\cdot e^{\int P(x)\,dx} &= \int e^{\int P(x)\,dx}Q(x)\,dx, \text{so} \\ y &= \cdot e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}Q(x)\,dx \end{aligned}$$

It is probably easier to memorize this process than to memorize this formula.

Examples
Solve each of the following differential equations:

1. $\frac{dy}{dx} = 4x - \frac{6y}{x}$
   Rearranging this, we get that $\frac{dy}{dx} + \frac{6y}{x} = 4x$, so $P(x) = \frac{6}{x}$ and our integrating factor is $\exp(\int \frac{6}{x}\,dx) = e^{6\ln|x|} = x^6$. (The "$+C$" can be ignored, since we can divide both sides by e^C and have an equivalent equation- see #27 for details). Multiplying through, we get that:

$$\begin{aligned} x^6\frac{dy}{dx} + x^6\frac{6y}{x} &= x^6 4x \\ x^6\frac{dy}{dx} + 6x^5y &= 4x^7 \\ \frac{d}{dx}\left(x^6y\right) &= 4x^7 \\ x^6y &= \int 4x^7\,dx = \frac{x^8}{2} + C \\ y &= \frac{x^2}{2} + Cx^{-6} \end{aligned}$$

2. $\frac{dy}{dx} - 2xy = x$ if $y = 1/2$ at $x = 0$
   We can let $P(x) = -2x$, so the integrating factor is $\exp(\int -2x\,dx) = e^{-x^2}$. Multiplying

through, we get:

$$
\begin{aligned}
e^{-x^2}\frac{dy}{dx} - e^{-x^2}2xy &= xe^{-x^2} \\
\frac{d}{dx}\left(e^{-x^2}y\right) &= xe^{-x^2} \\
e^{-x^2}y &= -\frac{e^{-x^2}}{2} + C \\
y &= e^{x^2}\left(-\frac{e^{-x^2}}{2} + C\right) = -\frac{1}{2} + Ce^{x^2}
\end{aligned}
$$

To find C, we use the initial condition, so we solve:

$$
\begin{aligned}
&\frac{1}{2} = -\frac{1}{2} + C \\
&C = 1
\end{aligned}
$$

Therefore, the particular solution is $y = -\frac{1}{2} + e^{x^2}$

3. A tank initially contains 200 gallons of brine, with 50 pounds of salt in solution. Brine containing 2 pounds of salt per gallon is entering the tank at the rate of 4 gallons per minute and is flowing out at the same rate. If the mixture in the tank is kept uniform by constant stirring, find the amount of salt in the tank at the end of 40 minutes. (#16)

   Let y represent the amount of salt in the tank at time t, where t is given in minutes. Then, $y = 50$ at $t = 0$ and

$$
\begin{aligned}
\frac{dy}{dt} &= \text{rate in} - \text{rate out} \\
\frac{dy}{dt} &= 8 - \frac{y}{50} \\
\frac{dy}{dt} + \frac{y}{50} &= 8 \\
e^{t/50}\frac{dy}{dt} + \frac{e^{t/50}}{50}y &= 8e^{t/50} \\
e^{t/50}y &= \int 8e^{t/50}\,dt = 400e^{t/50} + C \\
y &= 400 + Ce^{-t/50}. \text{ To find } C\text{, we use } y = 50 \text{ when } t = 0: \\
50 &= 400 + C, \text{ so } C = -350 \text{ and } y \text{ as a function of } t \text{ is}: \\
y &= 400 - 350e^{-t/50}
\end{aligned}
$$

   At $t = 40$, $y = 400 - 350e^{-40/50} = 242.735$ lbs of salt.

Kirchhoff's Law says that for a simple electrical current containing a resistor with a resistance of R ohms and an inductor with an inductance of L henrys in series with a source of electromotive force (a battery or generator) that supplies a voltage of $E(t)$ volts at time t satisfies:

$$L\frac{dI}{dt} + RI = E(t),$$

where I is the current measured in amperes.

Example
Find I as a function of time for the circuit with $R = 20\Omega$, $L = 5$ H, and $E(t) = 2$ volts, assuming that the switch is closed and $I = 0$ at $t = 0$.

We can set up (and solve) the differential equation from Kirchhoff's law by:

$$\begin{aligned}
5 \cdot \frac{dI}{dt} + 20I &= 2 \\
\frac{dI}{dt} + 4I &= \frac{2}{5} \\
e^{4t}\frac{dI}{dt} + 4e^{4t}I &= \frac{2e^{4t}}{5}. \text{ Integrating both sides, we get :} \\
e^{4t}I &= \frac{e^{4t}}{10} + C \\
I &= \frac{1}{10} + Ce^{-4t}. \text{ Using } I = 0 \text{ at } t = 0\text{, we get that} \\
I &= \frac{1}{10} - \frac{e^{-4t}}{10}
\end{aligned}$$