Section 6.5, Exponential Growth and Decay

Homework: 6.5 #1-27 odds, 37

Consider population growth. Let y represent the population at time t, and let y_0 be the population at the "start." Assuming that the rate of growth (or decay) is proportional to the size of the population is reasonable. This translates to $\frac{dy}{dt} = ky$. Solving this for y, we see that:

$$\begin{aligned}
dy &= ky\,dt \\
\int \frac{1}{y}\,dy &= \int k\,dt \\
\ln|y| &= kt + C, \text{ but } y \geq 0, \text{ so } |y| = y \\
\ln y &= kt + C \\
y &= e^{kt+C} \\
y &= e^{kt}e^{C} \text{ Letting } t = 0, \text{ we get that } e^{C} = y_0. \\
y &= y_0 e^{kt}
\end{aligned}$$

If $k > 0$, the population is growing. If $k < 0$, the population is decaying.

For population growth, note that there are other factors, such as the limiting capacity, L, so $\frac{dy}{dt} = ky(L - y)$ is more reasonable. However, for y small enough, $\frac{dy}{dt} \approx kLy$.

Examples

1. The population of the U.S. was 3.9 million in 1790 and 281.4 million in 2000. If the rate of grown is assumed proportion to the population, what estimate would you give for the population in 2010? Compare your answer to the actual population of 308.7 million.
   Let t represent the number of years since 1790, and let y be the population (in millions). Then, $y_0 = 3.9$, and k is the solution to

$$\begin{aligned}
281.4 &= 3.9 \cdot e^{210k} \\
\frac{281.4}{3.9} &= e^{210k} \\
\ln \frac{281.4}{3.9} &= 210k \\
k &= \frac{\ln \frac{281.4}{3.9}}{210} = 0.020375
\end{aligned}$$

   So, $y = 3.9e^{0.020375t}$. In 2010, $t = 220$, so our estimate for the population is $y = 3.9e^{0.020375 \cdot 220} = 345.0$. (There are many reasons that the estimate is off by almost 40 million, including the limiting capacity and the recession.)

2. If a radioactive substance loses 20% of its radioactivity in 7 days, what is its half-life? (i.e. the time until it has lost 50% of its radioactivity).
   Let t represent time in days. No initial amount is given, but we can assume $y_0 = 1$, since we are concerned with percentages. To find k, we need to use that $0.8 = e^{7k}$, so $k = (\ln 0.8)/7 \approx -0.03187765$. To find the half-life, we need to solve

$$\begin{aligned}
0.5 &= e^{-0.03187765 \cdot t} \\
-0.03187765 \cdot t &= \ln 0.5 \\
t &= \frac{\ln 0.5}{-0.03187765} \approx 21.744
\end{aligned}$$

Newton's Law of Cooling says that the rate at which an object cools (or warms) is proportional to the difference in temperature between the object and the surrounding medium. If $T(t)$ represents the temperature of the object at time t and T_1 is the temperature of the surrounding medium,

$$\frac{dT}{dt} = k(T - T_1)$$

Example
A batch of brownies is taken from a 350°F oven and placed in a refrigerator at 40°F and left to cool. After 15 minutes, the brownies have cooled to 250°F. When will the temperature of the brownies be 110°F? (#22)
We know that $T(0) = 350$ and $\frac{dT}{dt} = k(T - 40)$, so

$$\begin{aligned} \int \frac{dT}{T-40} &= \int k\,dt \\ \ln|T-40| &= kt + C. \quad \text{Since } T > 40\text{, we can eliminate the } |\cdot| \\ T - 40 &= T_1 e^{kt}. \quad \text{Since } T(0) = 350\text{, we can calculate } T_1 = 310 \\ T &= 310e^{kt} + 40 \end{aligned}$$

To find k, we need to use that $T(15) = 250$, so

$$\begin{aligned} 250 &= 310e^{15k} + 40 \\ 210 &= 310e^{15k} \\ \ln\frac{21}{31} &= 15k \\ k &= \frac{\ln\frac{21}{31}}{15} = -0.0259643178 \end{aligned}$$

so $T = 310e^{-0.02596\cdot t} + 40$. To find when $T = 110$, we solve

$$\begin{aligned} 110 &= 310e^{-0.02596\cdot t} + 40 \\ 70 &= 310e^{-0.02596\cdot t} \\ \ln\frac{7}{31} &= -0.02596\cdot t \\ t &= \frac{\ln\frac{7}{31}}{-0.02596} \approx 57.312 \text{ minutes} \end{aligned}$$

Theorem A $\lim_{h\to 0}(1+h)^{1/h} = e$.

Proof: Let $f(x) = \ln x$. Then,

$$1 = f'(1) = \lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\ln(1+h)^{1/h}$$

By exponentiating both sides, and using that e^x is continuous, we get:

$$e = e^1 = \exp\lim_{h\to 0}\ln(1+h)^{1/h} = \lim_{h\to 0}\exp\ln(1+h)^{1/h} = \lim_{h\to 0}(1+h)^{1/h}$$

Examples

1. Find $\lim_{h\to 0}(1-2h)^{1/h}$.

$$\begin{aligned} \lim_{h\to 0}(1-2h)^{1/h} &= \lim_{h\to 0}(1-2h)^{-2/-2h} \\ &= \left(\lim_{h\to 0}(1-2h)^{1/-2h}\right)^{-2} = e^{-2} \end{aligned}$$

2. Find $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$.

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e,$$

where the last step uses the substitution $h = 1/n$.

3. If \$500 is invested in an account earning 3%, compounded continuously, how much will be in the account after 4 years?
Interest compounded n times a year would result in $A = A_0(1+\frac{r}{n})^{nt}$. Since we want continuous compounding, $A = \lim_{n\to\infty} A_0(1+\frac{r}{n})^{nt} = A_0 e^{rt} = 500e^{0.03\cdot 4} = 563.75$.