Section 6.4, General Exponential and Logarithmic Functions

Homework: 6.4 #1–43 odds

Let $a \in \mathbb{R}$ such that $a > 0$ (and $a \neq 1$). Let $a^x = \exp(x \ln a)$ (Note that this implies that $\ln a^x = x \ln a$.). This is called the **exponential function with base** a.

Theorem A: Properties of Exponents

1. $a^x a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Proof of (4)**:**

$$\begin{aligned}
(ab)^x &= \exp\big(x \ln(ab)\big) \text{ by the definition of the exponential function with base } ab \\
&= \exp\big(x(\ln a + \ln b)\big) \\
&= \exp\big(x \ln a + x \ln b\big) \\
&= \exp\big(x \ln a\big) \exp\big(x \ln b\big) = a^x b^x
\end{aligned}$$

The proofs of the other statements are similar (and a few are in the book).

The **logarithm function with base** a, $\log_a x$, is the inverse of the exponential function with base a. In other words, $\log_a a^x = x$ and $a^{\log_a x} = x$. It can also be defined as $y = \log_a x \Leftrightarrow x = a^y$.

The **change of base formula** says that

$$\log_a x = \frac{\ln x}{\ln a}$$

To see this, let $y = \log_a x$, so $x = a^y$. Taking the natural logarithm of both sides and solving for y, we get the above formula.

1 Derivatives and Integrals

Let a be a positive real number. Then,

$$\begin{aligned}
&D_x a^x = a^x \ln a \\
&\int a^x \, dx = \frac{a^x}{\ln a} + C, \quad a \neq 1 \\
&D_x \log_a x = \frac{1}{x \ln a}, \quad a \neq 1
\end{aligned}$$

Be careful to not confuse the exponential formulas with the power rule.

We now have another way to show that the power rule holds for all real numbers a:

$$D_x(x^a) = D_x(e^{a \ln x}) = e^{a \ln x} \cdot \frac{a}{x} = x^a \cdot \frac{a}{x} = a x^{a-1}$$

Examples

1. Find y' if $y = 2^{\sin x}$.

   $$y' = 2^{\sin x}(\ln 2)(\cos x)$$

   Note: Be careful with the formatting of your answer, since $(\ln 2)(\cos x) \neq \ln(2\cos x)$.

2. Calculate $D_x(\log_5(4x^2 - 3))$.

   $$D_x(\log_5(4x^2 - 3)) = \frac{8x}{(4x^2 - 3)\ln 5}$$

3. Calculate $\int \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx$.
   Letting $u = \sqrt{x} = x^{1/2}$, $du = x^{-1/2}/2$, so

   $$\int \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int 3^u\,du = \frac{2 \cdot 3^u}{\ln 3} + C = \frac{2 \cdot 3^{\sqrt{x}}}{\ln 3} + C$$

For some functions, it is easier to first take the logarithm of both sides, then use implicit differentiation to calculate the derivative. This process is called **logarithmic differentiation**.

Example
Let $y = x^x$. Calculate y'.
We do not have derivative formulas for a variable raised to a variable power, so we need to manipulate the function. Taking the ln of both sides of the function, we get that:

$$\ln y = \ln x^x = x \ln x$$

$$\frac{y'}{y} = \ln x + \frac{x}{x} = \ln x + 1$$

$$y' = (\ln x + 1)y = (\ln x + 1)x^x$$

Note about the homework: Some of the problems involve solving logarithmic equations, which will properties of logarithms as well as manipulating expressions of the form $y = \log_a x$ to $x = a^y$.