Section 6.2, Inverse Functions and Their Derivatives

Homework: 6.2 #1–41 odds

Our goal for this section is to find a function that "undoes" a given function f by recovering the x-value that gave the y-value of the function. We will also look at some properties that it satisfies.

The function that "undoes" $f(x)$ is called the inverse of $f(x)$ and is denoted $f^{-1}(x)$. These functions satisfy $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ (Note: the domains may not be the same).

1 When does f^{-1} exist?

There are many criteria that can be checked to see if an inverse function exists. Some of them are:

- **One-to-one**: A function is called **one-to-one** if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. This implies that every y-value of the function comes from a unique value of x. This condition is equivalent to the geometric condition that every horizontal line meets the graph of $y = f(x)$ in at most one point.

  This condition can be difficult to check, since you need to know that the entire graph looks like.

- **Strict monotonicity**: A function that is either increasing or decreasing on its domain is called **strictly monotonic**. If a function is strictly monotonic on its domain, it has an inverse function (by Theorem A).

  This condition is normally easier to check than the one-to-one condition, since we can find $f'(x)$ and check that either $f'(x) > 0$ for every value of x, or $f'(x) < 0$ for every value of x.

Examples

1. Show that $f(x) = x^7 + 4x - 5$ has an inverse.
   We know that $f'(x) = 7x^6 + 4 > 0$, so the function is strictly increasing, which means that it has an inverse.

2. Does $f(x) = x^2 - 9$ have an inverse function? If not, can we restrict the domain so that it does?
   Since the graph of $y = f(x)$ is a parabola, we know that it is not one-to-one. Therefore, it does not have an inverse. However, if we consider just $x \geq 0$ or $x \leq 0$, it will have an inverse.

2 Finding inverse functions

Since we want to find a function that "undoes" $f(x)$, we can use that $x = f^{-1}(y)$ if and only if $y = f(x)$.

Steps for finding a formula for an inverse function:

1. Solve $y = f(x)$ for x in terms of y.
2. Switch the x and y variables.
3. Set $f^{-1}(x) = y$.

(Note: Using y instead of $f(x)$ in the first step is only used to make the notation easier when solving the equation.)

Example
Let $f(x) = \frac{3x-4}{x+1}$. Find $f^{-1}(x)$.
First, let $y = \frac{3x-4}{x+1}$. Solving for x, we get:

$$\begin{aligned} y(x+1) &= 3x-4 \\ yx + y &= 3x - 4 \\ yx - 3x &= -y - 4 \\ x(y-3) &= -y-4 \\ x &= \frac{-y-4}{y-3}, \end{aligned}$$

Therefore, $f^{-1}(x) = \frac{-x-4}{x-3}$. We can check our answer by verifying that $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(y) = y$.

If we want to graph $y = f^{-1}(x)$ and have information about the graph of $y = f(x)$, we can graph $y = f^{-1}(x)$ fairly easily. To do this, note that if (x, y) is a point on the graph of $y = f(x)$, then (y, x) is a point on the graph of $f^{-1}(x)$. This means that we can reflect the graph of $y = f(x)$ about the line $y = x$ and get the graph of $y = f^{-1}(x)$. (Also, the domain of $f(x)$ is the range of $f^{-1}(x)$ and the range of $f(x)$ is the domain of $f^{-1}(x)$.)

3 How do the derivatives compare?

If we know $f'(x)$, we want to be able to easily compute $(f^{-1})'(y)$ for $y = f(x)$.

Theorem B (the Inverse Function Theorem) says that if f is differentiable and strictly monotonic on an interval I, then f^{-1} is differentiable at the corresponding point $y = f(x)$ and

$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

This can also be written

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Example
Let $y = f(x) = x^7 + 4x - 5$. Find $(f^{-1})'(-5)$.
Note that $y = -5$ corresponds to $x = 0$. Also, $f'(x) = 7x^6 + 4$, so $(f^{-1})'(-5) = \frac{1}{f'(0)} = \frac{1}{4}$.