# Section 6.1, The Natural Logarithm Function

Homework: 6.1 #1–43 odds

The natural logarithm function, denoted by ln, is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

As a result,

$$D_x(\ln x) = \frac{1}{x}$$

**Examples**
Calculate $\frac{dy}{dx}$ for each of the following:

1. $y = \ln(3x)$

$$\frac{dy}{dx} = \frac{3}{3x} = \frac{1}{x}$$

2. $y = x\ln(x^2)$

$$\frac{dy}{dx} = \ln(x^2) + x \cdot \frac{2x}{x^2} = \ln(x^2) + 2$$

3. $y = \ln(2\sqrt{5x^3})$
We can rewrite $y = \ln(2(5x^3)^{1/2}) = \ln(2 \cdot 5^{1/2}x^{3/2})$

$$\frac{dy}{dx} = \frac{2 \cdot 5^{1/2} \cdot \frac{3}{2} \cdot x^{1/2}}{2 \cdot 5^{1/2}x^{3/2}} = \frac{3}{2x}$$

**Properties of Logarithms**

1. $\ln 1 = 0$
2. $\ln ab = \ln a + \ln b$
3. $\ln \frac{a}{b} = \ln a - \ln b$
4. $\ln a^r = r\ln a$

**Proof:**
(1): By the definition of the natural logarithm function, $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$.
(2): Let $a$ be a constant, and consider $f(x) = \ln(ax)$. Then, $f'(x) = \frac{a}{ax} = 1/x$. Since this function has the same derivative as $\ln x$, we know that $f(x) = \ln x + C$ for all $x > 0$. Using $x = 1$ and statement (1), we see that $\ln a = \ln 1 + C = C$, so $f(x) = \ln(ax) = \ln a + \ln x$.
(3): First, note that $0 = \ln 1 = \ln(b \cdot \frac{1}{b}) = \ln b + \ln \frac{1}{b}$, where the last equality is by statement (2). Rearranging this, we see that $\ln \frac{1}{b} = -\ln b$. Then, $\ln \frac{a}{b} = \ln(a \cdot \frac{1}{b}) = \ln a + \ln \frac{1}{b} = \ln a - \ln b$.
(4): Let $f(x) = \ln(x^r)$ for a constant $r$. Then, $f'(x) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}$. This has the same derivative as $r\ln x$, so $f(x) = r\ln x + C$. Using $x = 1$, we see that $C = 0$, so $f(x) = \ln(x^r) = r\ln x$.

**Examples**

1. Write the following as a single logarithm:

$$\ln x + 2\ln(x+1) - \ln(x+2)$$

$$\ln x + 2\ln(x+1) - \ln(x+2) = \ln x + \ln(x+1)^2 - \ln(x+2) = \ln \frac{x(x+1)^2}{x+2}$$

2. Find $dy/dx$ by logarithmic differentiation if $y = (x^2+1)^3(x-2)/(x+1)$.
Taking the natural logarithm of both sides, we see that $\ln y = 3\ln(x^2+1) + \ln(x-2) - \ln(x+1)$.
Taking the derivative of both sides with respect to $x$, we get that

$$\begin{aligned} \frac{dy/dx}{y} &= 3\frac{2x}{x^2+1} + \frac{1}{x-2} - \frac{1}{x+1}, \text{ so} \\ \frac{dy}{dx} &= \left(\frac{6x}{x^2+1} + \frac{1}{x-2} - \frac{1}{x+1}\right) \cdot y \\ &= \left(\frac{6x}{x^2+1} + \frac{1}{x-2} - \frac{1}{x+1}\right) \cdot \frac{(x^2+1)^3(x-2)}{x+1} \end{aligned}$$

**Integration**

$$\int \frac{1}{u}\,du = \ln|u| + C, \quad u \neq 0$$

Note that this fills in the "gap" in the power rule for integration ($\int x^n\,dx = \frac{x^{n+1}}{n+1}$ for $n \neq -1$).

**Examples**
Calculate each of the following integrals:

1. $\int \frac{x}{x^2-3}\,dx$

$$\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\ln\left|x^2-3\right| + C$$

2. $\int (3x-4)^{-1}\,dx$

$$\int (3x-4)^{-1}\,dx = \frac{1}{3}\ln\left|3x-4\right| + C$$

3. $\int \tan x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C$$

4. $\int \left[\ln(\sin x) + \ln(\csc x)\right] dx$

$$\int \left[\ln(\sin x) + \ln(\csc x)\right] dx = \int \left[\ln(\sin x) - \ln(\sin x)\right] dx = \int 0\,dx = C$$