Section 10.7, Calculus in Polar Coordinates

Homework: 10.7 #1–29 odds

1 Area

For a circle of radius r, note that the area of a sector ("wedge") of angle θ is

$$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}\theta r^2$$

Using this, we see that if $r = f(\theta)$, the area of the sector created by a curve from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[f(\theta)\right]^2 \, d\theta$$

Examples

1. Find the area inside the curve $r = 3 + 3\sin\theta$.

   We need to consider angles from 0 to 2π:

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} \left(3 + 3\sin\theta\right)^2 \, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(9 + 18\sin\theta + 9\sin^2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(9 + 18\sin\theta + \frac{9}{2}(1 - \cos(2\theta))\right) d\theta \\ &= \frac{1}{2}\left[9\theta - 18\cos\theta + \frac{9}{2}(\theta - \frac{\sin(2\theta)}{2})\right]\Bigg|_0^{2\pi} \\ &= \frac{1}{2}\left[9 \cdot 2\pi + \frac{9}{2} \cdot 2\pi\right] = \frac{27\pi}{2} \end{aligned}$$

2. Find the area inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

   After quickly graphing these, we can find the angles θ at which the graphs intersect:

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ 2\sin\theta &= 1 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

Therefore, to find the area, we can set up the difference of two integrals:

$$\begin{aligned}
A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 9\sin^2\theta\,d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(1 + 2\sin\theta + \sin^2\theta\right)d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(8\sin^2\theta - 2\sin\theta - 1\right)d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(4[1-\cos(2\theta)] - 2\sin\theta - 1\right)d\theta \\
&= \frac{1}{2}\left[3\theta - 2\sin(2\theta) + 2\cos\theta\right]\Bigg|_{\pi/6}^{5\pi/6} \\
&= \frac{1}{2}\left[\left(3\cdot\frac{5\pi}{6} - 2\cdot\frac{-\sqrt{3}}{2} + 2\cdot\frac{-\sqrt{3}}{2}\right) - \left(3\cdot\frac{\pi}{6} - 2\cdot\frac{\sqrt{3}}{2} + 2\cdot\frac{\sqrt{3}}{2}\right)\right] = \pi
\end{aligned}$$

2 Slope of the Tangent Line

In rectangular coordinates, we know that $m = \frac{dy}{dx}$. In Polar Coordinates,

$$y = r\sin\theta = f(\theta)\sin\theta \qquad \frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$$

$$x = r\cos\theta = f(\theta)\cos\theta \qquad \frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta,$$

so

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Examples

1. Find the slope of the line tangent to $r = 2 - 3\sin\theta$ at $\theta = \pi/6$.

   Note that $f(\theta) = 2 - 3\sin\theta$. First, let's calculate the formula for slope in polar coordinates:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(2-3\sin\theta)\cos\theta - 3\cos\theta\sin\theta}{-3\cos\theta\cos\theta - (2-3\sin\theta)\sin\theta} \\
&= \frac{2\cos\theta - 6\sin\theta\cos\theta}{-3\cos^2\theta - 2\sin\theta + 3\sin^2\theta} \\
&= \frac{2\cdot\frac{\sqrt{3}}{2} - 6\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}{-3\cdot\left(\frac{\sqrt{3}}{2}\right)^2 - 2\cdot\frac{1}{2} + 3\left(\frac{1}{2}\right)^2} \\
&= \frac{\sqrt{3} - \frac{3\sqrt{3}}{2}}{-\frac{5}{2}} = \frac{2\sqrt{3} - 3\sqrt{3}}{-5} = \frac{\sqrt{3}}{5}
\end{aligned}$$

2. For $r = 2 - 3\sin\theta$, when is the line tangent to the graph horizontal?

   Note that we calculated a formula for the slope of the line tangent to this curve in the last example. This means that we want to find where the formula equals zero.

$$\begin{aligned}
0 &= \frac{2\cos\theta - 6\sin\theta\cos\theta}{-3\cos^2\theta - 2\sin\theta + 3\sin^2\theta} \\
&= 2\cos\theta - 6\sin\theta\cos\theta \\
&= \cos\theta(2 - 6\sin\theta)
\end{aligned}$$

$$0 = \cos\theta \qquad\qquad 0 = 2 - 6\sin\theta$$

$$\theta = \frac{\pi}{2} + n\pi \qquad\qquad \sin\theta = \frac{1}{3}$$

$$\theta = \sin^{-1}\frac{1}{3} + 2n\pi,\ \pi - \sin^{-1}\frac{1}{3} + 2n\pi$$