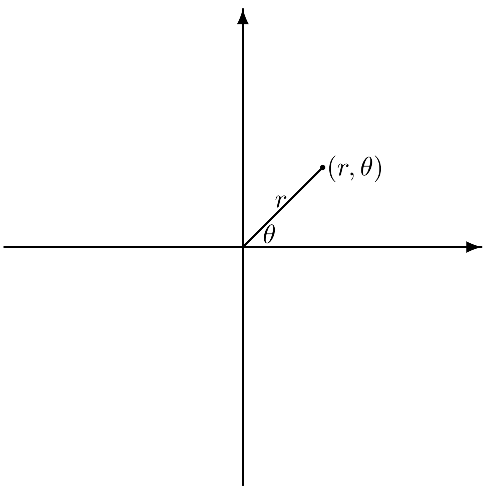
Section 10.5, The Polar Coordinate System

Homework: 10.5 #1–35 odds

So far this semester, we have focused on the rectangular coordinate system (or Cartesian system) that includes points of the form (x, y). We will now be focusing on a different way to graph points in a plane, called the **Polar Coordinate System**.



We call the origin the **pole**, and the **polar axis** is equivalent to the positive x-axis.

1 Converting Between Polar and Rectangular Coordinates

To convert between rectangular and polar coordinates, we can use the following formulas:

$$x = r\cos\theta \qquad \tan\theta = \frac{y}{x}$$

$$y = r\sin\theta \qquad r^2 = x^2 + y^2 \text{ or } r = \sqrt{x^2 + y^2}$$

Examples

1. Convert $(3, \frac{\pi}{2})$ to rectangular coordinates.

   We can use $x = r\cos\theta$ and $y = r\sin\theta$ to get values for x and y:

   $$x = r\cos\theta = 3\cos\frac{\pi}{2} = 0$$
   $$y = r\sin\theta = 3\sin\frac{\pi}{2} = 3,$$

   so the point in rectangular coordinates is $(0, 3)$.

2. Convert $(-1, -1)$ to polar coordinates.

   First, we can find r:

   $$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

   To find θ, we can use that the point is in Quadrant III along with $\tan\theta = y/x$ to get

   $$\tan\theta = \frac{-1}{-1} = 1$$
   $$\theta = \frac{5\pi}{4},$$

so the point is $(\sqrt{2}, \frac{5\pi}{4})$.

3. Find 4 other ways to write the $(-3, 2\pi/3)$.

   We can use any coterminal angle to represent the same point, so we can use:

   $(-3, -4\pi/3)$, $(-3, 8\pi/3)$

   We can also eliminate the negative radius by adding π (180°) to the angle , so we get

   $(3, 5\pi/3)$, $(3, 11\pi/3)$

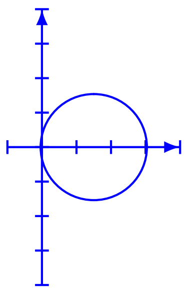
   , where the second point uses an angle that is coterminal with the first point.

4. Sketch the graph of $r = 3\cos\theta$. Then show that $r = 3\cos\theta$ is a circle in the Cartesian coordinate system.

   Similarly to graphing in rectangular coordinates, we can use several values of θ to see what value is given for r, then graph the function:

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$3\pi/2$	$11\pi/6$	2π
r	3	$3\sqrt{3}/2$	$3/2$	0	$-3/2$	$-3\sqrt{3}/2$	-3	$-3\sqrt{3}/2$	0	$3\sqrt{3}/2$	3

   Since cosine is 2π-periodic, we only need to consider θ between 0 and 2π (or any other interval with length 2π). Graphing the information that we found, we get the graph:



   Note: This gets traced twice when graphing between $\theta = 0$ and 2π.

   To rigorously show that $r = 3\cos\theta$ is a circle, we can change variables to the Cartesian coordinate system, then verify that it has the form $(x-a)^2 + (y-b)^2 = c^2$.

$$
\begin{aligned}
r &= 3\cos\theta \\
\sqrt{x^2+y^2} &= 3\frac{x}{r} \\
\sqrt{x^2+y^2} &= 3\frac{x}{\sqrt{x^2+y^2}} \\
x^2+y^2 &= 3x \\
x^2-3x+y^2 &= 0 \quad \text{Completing the square, we get} \\
x^2-3x+\frac{9}{4}+y^2 &= \frac{9}{4} \\
\left(x-\frac{3}{2}\right)^2+y^2 &= \frac{9}{4}
\end{aligned}
$$

   which is a circle of radius $\sqrt{9/4} = 3/2$ centered at the point $(3/2, 0)$. This agrees with our graph.

2 Polar Equations

2.1 Lines

Let $P = (r, \theta)$ represent a point on the line, d be the shortest distance from the origin to the line, and θ_0 be the angle from the polar axis to the closest point on the line. Then, we can sketch a triangle and see that:

$$\cos(\theta_0 - \theta) = \frac{d}{r} \quad \text{or}$$
$$r = \frac{d}{\cos(\theta_0 - \theta)} = \frac{d}{\cos(\theta - \theta_0)},$$

since cosine is an even function.

2.2 Circles

Let $P = (r, \theta)$ represent a point on the circle, a be the radius of the circle, c be the shortest distance from the origin to the center of the circle, and θ_0 be the angle from the polar axis to the line to the center of the circle. Then, we can sketch a triangle and see that

$$a^2 = r^2 + c^2 - 2rc\cos\big(\theta - \theta_0\big)$$

by the Law of Cosines.

If $c = a$, the circle passes through the origin and the equation simplifies to

$$r = 2a\cos\big(\theta - \theta_0\big)$$

If $\theta_0 = 0$, the center of the circle is on the x-axis, and the equation simplifies to $r = 2a\cos(\theta)$.

If $\theta_0 = \pi/2$, the center of the circle is on the y-axis, and the equation simplifies to $r = 2a\sin(\theta)$.

2.3 Conics: Parabolas, Hyperbolas, and Ellipses

Assume that the focus F is at the origin, and let $P = (r, \theta)$ be a point on the conic. Let ℓ be a line. The **eccentricity** e is the ratio of the distance from P to F to the distance from P to ℓ. Also let d be the distance from F to the line ℓ and θ_0 be the angle of that direction. Then,

$$|PF| = e|P\ell|$$

where $|PQ|$ is the distance between points P and Q. Then, if a is the length of the side of the right triangle including points F and P along the shortest line to ℓ,

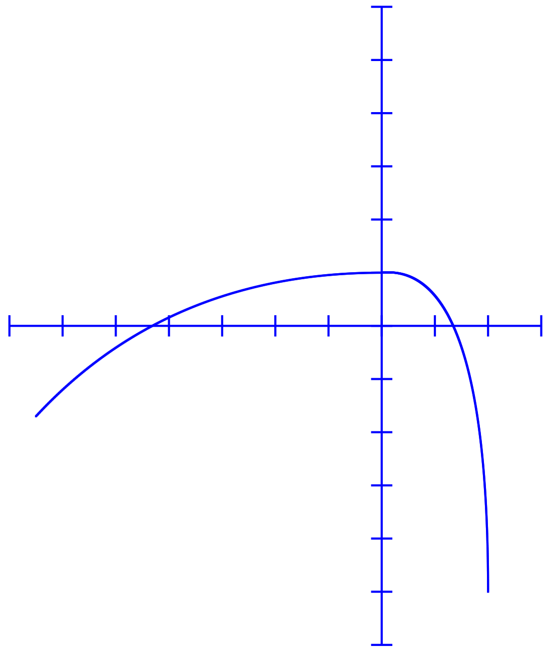
$$r = e|PL|,$$
$$\cos(\theta - \theta_0) = \frac{a}{r} \quad \text{so} \quad a = r\cos(\theta - \theta_0), \text{ and}$$
$$|P\ell| = d - a = d - r\cos(\theta - \theta_0) \quad \text{Then, the equation becomes}$$
$$r = e(d - r\cos(\theta - \theta_0)) \quad \text{Solving for } r\text{, we see that}$$
$$r = \frac{ed}{1 + e\cos(\theta - \theta_0)}$$

If $0 < e < 1$, the graph is an ellipse. If $e = 1$, this is the equation of a parabola. If $e > 1$, the graph is a hyperbola.

Examples
Name the curve with the given polar equation. If it is a conic, give its eccentricity. Sketch the graph.

1. $r = \dfrac{4}{2 + 2\cos(\theta - \pi/3)}$

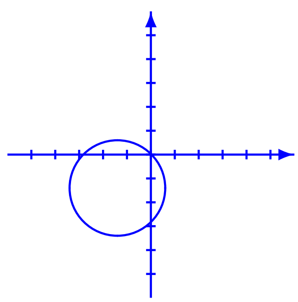
   We can first reduce the fraction to become $r = \frac{2}{1+\cos(\theta-\pi/3)}$, so we see that this is a conic with eccentricity 1 (in other words, the graph will be a parabola). Then, the graph is



This is the graph of a parabola, but it is "tilted" from how we often see them in algebra classes ($y = ax^2 + bx + c$).

2. $r = -4\cos(\theta - \pi/4)$

   This is the equation of a circle of radius 2 through the origin. We can rewrite the equation as $r = 4\cos(\theta - 5\pi/4)$, so this is a circle centered on the line $\theta = 5\pi/4$:



3. $\theta = \dfrac{2\pi}{3}$

   This is the equation of a line: