

1. Introduction

I study nonlinear partial differential equations. More specifically, I have worked on elliptic problems, most of which are related to the p -Laplace operator. In this brief note, I will provide an overview of my current work, describe my plans for continuing research, and introduce some problems that interest me.

2. Nonlinear Eigenvalue Problems

Many eigenvalue problems can be formulated as extremal problems with side conditions, i.e. finding critical points of a given functional with respect to a given manifold or a level set. A classical example in partial differential equations is to find eigenvalues of the Laplace operator in a smooth bounded domain Ω with Dirichlet boundary condition:

$$\begin{cases} \Delta u + \lambda u &= 0, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega. \end{cases} \quad (1)$$

This problem leads to finding critical points of the functional $F(u) := \int_{\Omega} |\nabla u|^2 dx$ defined in the Sobolev space $W_0^{1,2}(\Omega)$ restricted to the level set $M := \{u \in W^{1,2}(\Omega) : G(u) := \int_{\Omega} |u|^2 dx = 1\}$.

The Lagrange multiplier principle (or Euler-Lagrange equation) enables one to find an eigenvalue, the first eigenvalue or the principal eigenvalue. To obtain higher eigenvalues of (1), one uses the Courant max-min principle (cf. [6, 11]). This powerful principle may be applied to many linear eigenvalue problems, for example in a similar way one may capture the entire spectrum of a compact symmetric operator in some Hilbert space.

In recent decades, nonlinear analysis has played a very important role in the application of mathematics to practical problems. Consequently eigenvalue problems for quasilinear (or fully nonlinear) differential operators subject to boundary conditions have emerged as significant mathematical problems. Such problems are, of course, also of interest for mathematical reasons only. My interest is in studying nonlinear eigenvalue problems which are given by nonlinear partial differential equations subject to boundary conditions, most of which involve the p -Laplace operator ($\Delta_p := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $p > 1$).

The p -Laplace operator arises in the theory of non-Newtonian fluids, in reaction-diffusion problems, in flow through porous media, in nonlinear elasticity, in petroleum extraction ([8]), in torsional creep problems ([5]), to name a few. Many results have been obtained on the structure of its spectrum subject to Dirichlet boundary conditions:

$$\text{Dirichlet: } \begin{cases} \Delta_p u + \lambda |u|^{p-2} u &= 0, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega, \end{cases} \quad (2)$$

where Ω is a bounded domain in \mathbb{R}^N with C^1 or Lipschitz continuous boundary. An analog of the Courant max-min principle in nonlinear eigenvalue problems is the Ljusternik-Schnirelman (L-S) principle. Using the L-S principle together with regularity theory one can show that:

- (i) there exists a nondecreasing sequence of positive eigenvalues, [12];
- (ii) all eigenvectors are $C^{1,\alpha}(\bar{\Omega})$ functions, [9, 10, 18];
- (iii) the principal (smallest) eigenvalue is simple and isolated, [1, 19];
- (iv) the set of eigenvalues is closed;
- (v) a characterization of the second eigenvalue is given, [2].

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When I was studying the Dirichlet problem (2), Professor Klaus Schmitt (my Ph.D. advisor) asked me to characterize the spectrum if other conditions, such as Neumann, Robin and No-flux conditions are imposed:

$$\begin{aligned}
\text{Neumann:} & \quad \begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega, \end{cases} \\
\text{Robin:} & \quad \begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} + \beta |u|^{p-2} u = 0, & \text{on } \partial\Omega, \beta > 0, \end{cases} \\
\text{No-flux:} & \quad \begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0, & \text{in } \Omega, \\ u = \text{constant} & \text{on } \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} = 0. \end{cases}
\end{aligned}$$

As mentioned at the beginning of the section, we can obtain eigenvalues of these problems by formulating appropriate extremal problems. In order to apply the L-S principle we need to find corresponding functionals and suitable function spaces. Instead of studying each problem above separately, one may employ the same functionals and determine the function space by the boundary condition. Thus one becomes interested in the existence of critical points of

$$F(u) := \int_{\Omega} |u|^p dx \text{ restricted to the level set } M = G^{-1}\{0\}, \quad G(u) := \int_{\Omega} (|\nabla u|^p + |u|^p) dx,$$

in various function spaces such as $W_0^{1,p}(\Omega)$, $W^{1,p}(\Omega)$ and $W^{1,p}(\Omega) \oplus \mathbb{R}$ which correspond to the Dirichlet problem, the Neumann problem, and the No-flux problem, respectively.

With this general setting, in my thesis [16] and in [15] I unified those three problems and then showed that their spectra share a similar structure (assertions (i)-(v) on the previous page). Furthermore, by modifying the functional G , the Robin problem can be included as well and thus I obtain the same results for the structure of its spectrum. This way of looking at partial differential equations with boundary conditions has allowed me to extend the earlier results ((i)-(v)) to other nonlinear eigenvalue problems. One of which is the generalized Steklov problem (see [20]),

$$\text{Steklov: } \begin{cases} \Delta_p u = |u|^{p-2} u, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} = \lambda |u|^{p-2} u, & \text{on } \partial\Omega. \end{cases}$$

The others are degenerate eigenvalue problems for the weighted p -Laplace operator, such as:

$$\begin{cases} -\operatorname{div}((a(x)|\nabla u|^{p-2}\nabla u)) = \lambda b(x)|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (3)$$

In the latter problems, we apply the L-S principle in weighted Sobolev spaces to establish the existence of sequences of eigenvalues of (3) (for technical details, see [17]).

3. Future Plans

It is my intention to continue the study of such (and more general) nonlinear eigenvalue problems which can be formulated and studied this way. Besides that there are several interesting problems arising in the nonlinear case that I want to pursue in the future:

1. Eventhough the first and second eigenvalue have been completely described, it is not known if the L-S sequence accounts for the entire spectrum. In particular, it has not been proved that it is impossible that every $\lambda \geq \lambda_2$ is an eigenvalue.

2. Since the problem is nonlinear, it is not clear how to define multiplicities of higher (than the first) eigenvalue.

3. What type of Fredholm alternatives may be derived for such eigenvalues? And what type of resonance or nonresonance results?

We know that for the Dirichlet problem (1) or (2) much is known about these questions in the linear case $p = 2$ (see [6, 7, 22]) or in the one dimensional case $N = 1$ (see [21]).

During my first two months at the Mathematical Sciences Research Institute, Berkeley, California, I have been exposed to (among other interesting topics) the study of the ∞ -Laplace operator Δ_{∞} , where

$\Delta_\infty u = \sum u_i u_j u_{ij}$. Motivated by the work of Aronsson ([3, 4]) concerning extensions of functions satisfying Lipschitz conditions on the boundary of a given domain Ω , the study of the ∞ -Laplace operator can be considered as the limiting case of the p -Laplace operator as $p \rightarrow \infty$, cf. [5]. Regarding the existence of eigenvalues, it is proved in [13, 14] that one can obtain a sequence of eigenvalues $\{\Lambda_k\}$ of the ∞ -Laplace operator as the limit of a given sequence $\{\lambda_k(p)\}$ of the p -Laplace operator by letting $p \rightarrow \infty$, i.e. for each k

$$\lim_{p \rightarrow \infty} [\lambda_k(p)]^{1/p} = \Lambda_k, \quad (4)$$

where $\lambda_k(p)$ is the k -th L-S eigenvalue of p -Laplace operator. As a consequence, many properties of the spectrum of the p -Laplace operator survive the passage to the limit. An interesting fact about $\{\Lambda_k\}$ is that

$$\frac{1}{\Lambda_k} \leq \sup\{r : \text{there are } k \text{ disjoint balls of radius } r \text{ inside } \Omega\}, \quad (5)$$

and the equality holds for the first and second eigenvalue, $k = 1$ and $k = 2$.

The limit characterization (4) and the geometric characterization (5) provide us two ways to analyze the spectrum of the ∞ -Laplace operator. The more we know about the p -Laplace operator and its spectrum the more information we have about the ∞ -Laplace operator. On the other hand, understanding the geometric characterization of $\{\Lambda_k\}$ will allow us to say more about the p -Laplace operator.

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