EIGENVALUE PROBLEMS FOR THE p-LAPLACIAN

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Abstract. We study nonlinear eigenvalue problems for the p-Laplace operator subject to different kinds of boundary conditions on a bounded domain. Using the Ljusternik-Schnirelman principle, we show the existence of a nondecreasing sequence of nonnegative eigenvalues. We prove the simplicity and isolation of the principal eigenvalue and give a characterization for the second eigenvalue.

1. Introduction

Eigenvalue problems for the $p-$Laplace operator subject to zero Dirichlet boundary conditions on a bounded domain have been studied extensively during the past two decades and many interesting results have been obtained. The investigations principally have relied on variational methods and deduce the existence of a *principal* eigenvalue as a consequence of minimization results of appropriate functionals. This principal eigenvalue then is the smallest of all possible eigenvalues and its existence proof is very much the same for all possible types of boundary conditions. The study of higher eigenvalues, on the other hand, introduces complications which depend upon the boundary conditions in a significant way, and thus the existence proofs may differ significantly, as well. On the other hand, there is a large class of commonly studied eigenvalue problems which allow for a unified treatment. It is such a class of problems which is being studied here. We consider, among others, the following eigenvalue problems:

- Dirichlet problem:

$$D(\Omega):\begin{cases} -\Delta_p u &= \lambda|u|^{p-2}u, \text{ in } \Omega,\\ u &= 0, \text{ on } \partial\Omega.\end{cases}$$

- No-flux problem:

$$P(\Omega):\begin{cases} -\Delta_p u &= \lambda|u|^{p-2}u, \text{ in } \Omega,\\ u &= \text{ constant }, \text{ on } \partial\Omega,\\ \int_{\partial\Omega}|\nabla u|^{p-2}\frac{\partial u}{\partial n}ds &= 0.\end{cases}$$

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- Neumann problem:

$$N(\Omega):\begin{cases} -\Delta_p u &= \lambda|u|^{p-2}u, \text{ in } \Omega,\\ \frac{\partial u}{\partial n} &= 0, \text{ on } \partial\Omega.\end{cases}$$

- Robin problem:

$$R(\Omega):\begin{cases} -\Delta_p u &= \lambda|u|^{p-2}u, \text{ in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial n}+\beta|u|^{p-2}u &= 0, \text{ on } \partial\Omega.\end{cases}$$

- Steklov problem:

$$S(\Omega):\begin{cases} \Delta_p u &= |u|^{p-2}u, \text{ in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} &= \lambda|u|^{p-2}u, \text{ on } \partial\Omega.\end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $p>1$, and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u with respect to $\partial\Omega$. We note that, when $N=1$ and $\Omega=(a,b)$, $P(\Omega)$ becomes the periodic boundary value problem

$$\begin{cases} -(|u'|^{p-2}u')' &= \lambda|u|^{p-2}u, \text{ in } (a,b),\\ u(a) &= u(b),\\ u'(a) &= u'(b).\end{cases}$$

The parameter β (which may be a function) in $R(\Omega)$ is in $[0,\infty)$. We observe that the Dirichlet and Neumann problems correspond to the cases $\beta=0$ and $\beta=\infty$, respectively.

Besides being of mathematical interest, the study of the p-Laplacian operator is also of interest in the theory of non-Newtonian fluids both for the case $p\geq 2$ (dilatant fluids) and $1<p<2$ (pseudo-plastic fluids), see [4]. It is also of geometrical interest for $p\geq 2$, some of which is discussed in [32].

Many results have been obtained on the structure of the spectrum of the Dirichlet problem $D(\Omega)$. It is shown in [17] that there exists a nondecreasing sequence of positive eigenvalues $\{\lambda_n\}$ tending to ∞ as $n\to\infty$. Moreover, the first eigenvalue is simple and isolated, see [2, 23]. Recently, in [3], a characterization of the second eigenvalue of $D(\Omega)$ was also given.

The existence of such a sequence of eigenvalues can be proved using the theory of Ljusternik - Schnirelman (e.g. see [7, 16, 35]). For that reason we call this sequence **the L-S sequence** $\{\lambda_n\}$. To establish the simplicity of the first eigenvalue λ_1, one shows that any (nontrivial) eigenfunction associated to λ_1 does not change sign and that any two first eigenfunctions are constant multiples of each other. It follows from the proof of the simplicity of λ_1 that any eigenfunction associated to an eigenvalue $\lambda\neq\lambda_1$ has to change sign. This fact together with the closedness of the spectrum give the isolation of λ_1. It will also be shown that the eigenvalue λ_2 of the L-S sequence is in fact the least of all eigenvalues which exceed the first eigenvalue.

The spectrum of the Dirichlet problem $D(\Omega)$ has, among others, the following properties:

(i) There exists a nondecreasing sequence of nonnegative eigenvalues obtained by the Ljusternik-Schnirelman principle (L-S sequence) $\{\lambda_n\}$ tending to ∞ as $n \to \infty$ (García Azorero and Peral Alonso [17]).
(ii) The first eigenvalue λ_1 is simple and only eigenfunctions associated to λ_1 do not change sign (Anane [2] and Lindqvist [23]).
(iii) The set of eigenvalues is closed (Anane and Tousli [3]).
(iv) The first eigenvalue λ_1 is isolated (Lindqvist [23]).
(v) The eigenvalue λ_2 is the second eigenvalue ([3]), i.e.,

$$\lambda_2 = \inf\{\lambda : \lambda \text{ is an eigenvalue of } D(\Omega) \text{ and } \lambda > \lambda_1\}.$$

Comparatively, the spectra of $N(\Omega)$, $P(\Omega)$, $R(\Omega)$, $S(\Omega)$ have been investigated little and it is natural to pose the problem of analyzing the structures of the spectra of $N(\Omega)$, $P(\Omega)$, $R(\Omega)$, $S(\Omega)$ and compare these to that of $D(\Omega)$.

We remark that in the case $N = 1$ fairly complete information on the spectra of the Dirichlet, Neumann, and periodic boundary value problems is available (see, e.g., [25], [14]).

The purpose of this paper is to study this problem and show, among other things, that properties (i)-(v) of the spectrum of the Dirichlet problem also hold for $P(\Omega)$, $N(\Omega)$, $R(\Omega)$ and $S(\Omega)$.

By choosing appropriate function spaces, we will see later in section 2 that we can unify all problems as one single abstract problem. And if we denote by $\{\lambda_n^D\}$, $\{\lambda_n^P\}$, $\{\lambda_n^N\}$ the corresponding L-S sequences of eigenvalues of $D(\Omega)$, $P(\Omega)$, $N(\Omega)$ (respectively) then

$$\lambda_n^D \geq \lambda_n^P \geq \lambda_n^N, \text{ for all } n. \tag{1.1}$$

The paper is organized as follows: We first present, for the sake of completeness the Ljusternik-Schnirelman principle and applications to our setting. We then establish the existence of L-S sequences for the Dirichlet, Periodic, Neumann, Robin, and Steklov problems. This is followed by a discussion of global boundedness and $C^{1,\alpha}$ smoothness of eigenfunctions. In Section 5 we establish the promised properties of the spectra of the Dirichlet, Periodic, Neumann, Robin and Steklov problems. The final section consists of some appendices containing useful results which are needed in the development.

2. The Ljusternik-Schnirelman principle and its applications

2.1. **The Ljusternik-Schnirelman principle in Banach spaces.** We recall here a version of the Ljusternik-Schnirelman principle which was discussed by F. Browder [7] and E. Zeidler [34], [35] (section 44.5, remark 44.23). We then shall apply the principle (theorem 2.1) to establish the existence of a sequence of eigenvalues for eigenvalue problems in closed subspaces of $W^{1,p}(\Omega)$.

Let X be a real reflexive Banach space and F, G be two functionals on X. Consider the following eigenvalue problem

$$F'(u) = \mu G'(u),\ u \in S_G,\ \mu \in \mathbb{R}, \tag{2.1}$$

where S_G is the level $S_G = \{u \in X : G(u) = 1\}$.

We assume that:

(H1) $F, G : X \to \mathbb{R}$ are even functionals and that $F, G \in C^1(X, \mathbb{R})$ with $F(0) = G(0) = 0$.

(H2) F' is strongly continuous (i.e. $u_n \rightharpoonup u$ in X implies $F'(u_n) \to F'(u)$) and $\langle F'(u), u\rangle = 0,\ u \in \overline{coS_G}$ implies $F(u) = 0$, where $\overline{coS_G}$ is the closed convex hull of S_G.

(H3) G' is continuous, bounded and satisfies condition (S_0), i.e. as $n \to \infty$,

$$u_n \rightharpoonup u,\ G'(u_n) \rightharpoonup v,\ \langle G'(u_n), u_n\rangle \to \langle v, u\rangle \text{ implies } u_n \to u.$$

(H4) The level set S_G is bounded and $u \neq 0$ implies

$$\langle G'(u), u\rangle > 0,\ \lim_{t \to +\infty} G(tu) = +\infty,\ \inf_{u \in S_G} \langle G'(u), u\rangle > 0.$$

It is known that (u, μ) solves (2.1) if and only if u is a critical point of F with respect to S_G (see Zeidler [35], proposition 43.21).

For any positive integer n, denote by $\mathbb{A}_n$ the class of all compact, symmetric subsets K of S_G such that $F(u) > 0$ on K and $\gamma(K) \geq n$, where $\gamma(K)$ denotes the genus of K, i.e., $\gamma(K) := \inf\{k \in \mathbb{N} : \exists h : K \to \mathbb{R}^k \setminus \{0\}$ such that h is continuous and odd $\}$.

We define:

$$a_n = \begin{cases} \sup_{H \in \mathbb{A}_n} \inf_{u \in H} F(u), & \mathbb{A}_n \neq \emptyset. \\ 0, & \mathbb{A}_n = \emptyset. \end{cases} \tag{2.2}$$

Also let

$$\chi = \begin{cases} \sup\{n \in \mathbb{N} : a_n > 0\}, & \text{if } a_1 > 0, \\ 0, & \text{if } a_1 = 0. \end{cases} \tag{2.3}$$

We now state the Ljusternik-Schnirelman principle.

Theorem 2.1. *Under assumptions (H1)-(H4), the following assertions hold:*

1. *If $a_n > 0$, then (2.1) possesses a pair $\pm u_n$ of eigenvectors and an eigenvalue $\mu_n \neq 0$; furthermore $F(u_n) = a_n$.*
2. *If $\chi = \infty$, (2.1) has infinitely many pairs $\pm u$ of eigenvectors corresponding to nonzero eigenvalues.*
3. *$\infty > a_1 \geq a_2 \geq ... \geq 0$ and $a_n \to 0$ as $n \to \infty$.*
4. *If $\chi = \infty$ and $F(u) = 0,\ u \in \overline{coS_G}$ implies $\langle F'(u), u\rangle = 0$, then there exists an infinite sequence $\{\mu_n\}$ of distinct eigenvalues of (2.1) such that $\mu_n \to 0$ as $n \to \infty$.*
5. *Assume that $F(u) = 0,\ u \in \overline{coS_G}$ implies $u = 0$. Then $\chi = \infty$ and there exists a sequence of eigenpairs $\{(u_n, \mu_n)\}$ of (2.1) such that $u_n \rightharpoonup 0,\ \mu_n \to 0$ as $n \to \infty$ and $\mu_n \neq 0$ for all n.*

Proof. We refer to [7] or [34] for the proof. □

2.2. **Applications of the L-S principle in $W^{1,p}(\Omega)$ and its subspaces.** Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary. Let X be a closed subspace of $W^{1,p}(\Omega)$ such that $W_0^{1,p}(\Omega) \subseteq X \subseteq W^{1,p}(\Omega)$ with the norm $\|\cdot\|$ induced by the norm in $W^{1,p}(\Omega)$. Define on X the functionals

$$F(u) = \int_\Omega a(x)|u(x)|^p dx + \int_{\partial\Omega} b(s)|u(s)|^p ds, \tag{2.4}$$

$$G(u) = \int_\Omega (|\nabla u(x)|^p + |u(x)|^p) dx + \int_{\partial\Omega} \beta(s)|u(s)|^p ds, \tag{2.5}$$

where $a \in L^\infty(\Omega)$ and $b,\ \beta \in L^\infty(\partial\Omega)$ such that $a,\ b,\ \beta \geq 0$ a.e. (We refer to [20] where the the surface integral on $\partial\Omega$ and the spaces $L^p(\partial\Omega)$ are discussed.)

As before we define $S_G = \{u \in X : G(u) = 1\}$.

It is easy to see that F and G are C^1 functionals. Let

$$A = \frac{1}{p}F',\ B = \frac{1}{p}G',$$

where

$$\langle Au, v\rangle = \int_\Omega a|u|^{p-2}uv dx + \int_{\partial\Omega} b|u|^{p-2}uv ds, \tag{2.6}$$

$$\langle Bu, v\rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv) dx + \int_{\partial\Omega} \beta|u|^{p-2}uv ds, u, v \in X. \tag{2.7}$$

Then (2.1) becomes $Au = \mu Bu$, where $G(u) = 1$. Thus for any $v \in X$,

$$\int_\Omega a|u|^{p-2}uv dx + \int_{\partial\Omega} b|u|^{p-2}uv ds = \mu\left(\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv) dx + \int_{\partial\Omega} \beta|u|^{p-2}uv ds\right). \tag{2.8}$$

We claim that F, G satisfy hypotheses (H1), (H2), (H3), and (H4) mentioned in 2.1.

It follows straightforwardly from (2.6), (2.7) that (H1) and (H4) hold.

Proposition 2.2. *Let F be defined in (2.4), then F' satisfies (H2).*

Proof. It suffices to show that A is strongly continuous. Let $u_n \rightharpoonup u$ in X, we need to show that $Au_n \to Au$ in X^*.

For any $v \in X$, by Hölder's inequality in the space $L^p(\partial\Omega)$ and Sobolev's embedding theorem, it follows that

$$\begin{aligned}
|\langle Au_n - Au, v\rangle| &\leq \left|\int_\Omega a(|u_n|^{p-2}u_n - |u|^{p-2}u)vdx\right| + \\
&\quad \left|\int_{\partial\Omega} b(|u_n|^{p-2}u_n - |u|^{p-2}u)vds\right| \\
&\leq \|a\|_\infty \||u_n|^{p-2}u_n - |u|^{p-2}u\|_{\frac{p}{p-1}} \|v\|_p + \\
&\quad \|b\|_\infty \||u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{p}{p-1}}(\partial\Omega)} \|v\|_{L^p(\partial\Omega)} \\
&\leq C_1\|a\|_\infty \||u_n|^{p-2}u_n - |u|^{p-2}u\|_{\frac{p}{p-1}} \|v\| + \\
&\quad C_2\|b\|_\infty \||u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{p}{p-1}}(\partial\Omega)} \|v\|,
\end{aligned}$$

where we have denoted (and we shall continue to do so) by $\|\cdot\|$, $\|\cdot\|_p$ the norms in $W^{1,p}(\Omega)$, and $L^p(\Omega)$, respectively.

We next show $|u_n|^{p-2}u_n \to |u|^{p-2}u$ in $L^{\frac{p}{p-1}}(\Omega)$. To see this, let $w_n = |u_n|^{p-2}u_n$ and $w = |u|^{p-2}u$. Since $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$, it follows

$$w_n(x) \to w(x), \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega |w_n|^{\frac{p}{p-1}} dx \to \int_\Omega |w|^{\frac{p}{p-1}} dx.$$

We conclude from lemma A.1 that $w_n \to w$ in $L^{p/(p-1)}(\Omega)$.

Using the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ and arguing as above, we obtain that $|u_n|^{p-2}u_n \to |u|^{p-2}u$ in $L^{\frac{p}{p-1}}(\partial\Omega)$. Therefore $Au_n \to Au$ in X^*. □

In order to verify (H3) we need the following lemma which uses a calculation from chapter 6 of [21].

Lemma 2.3. *Let B be defined in (2.7), then for any $u, v \in X$ one has*

$$\langle Bu - Bv, u - v\rangle \geq (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).$$

Furthermore, $\langle Bu - Bv, u - v\rangle = 0$ if and only if $u = v$ a.e. in Ω.

Proof. Straightforward computations give us

$$\begin{aligned}
&\langle Bu - Bv, u - v\rangle \\
= &\int_\Omega [|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla v - |\nabla v|^{p-2}\nabla v \cdot \nabla u]dx \\
&+ \int_\Omega (|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu)dx \\
&+ \int_{\partial\Omega} \beta(|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu)ds.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\partial\Omega} \beta(|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu)ds &\geq \\
\int_{\partial\Omega} \beta(|u|^p + |v|^p - |u|^{p-1}|v| - |v|^{p-1}|u|)ds &= \\
\int_{\partial\Omega} \beta(|u|^{p-1} - |v|^{p-1})(|u| - |v|)ds &\geq 0,
\end{aligned}$$

we have

$$\begin{aligned}
& \langle Bu - Bv, u - v\rangle \\
\geq\ & \int_\Omega [|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla v - |\nabla v|^{p-2}\nabla v \cdot \nabla u]dx \\
& + \int_\Omega (|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu)dx \\
=\ & \|u\|^p + \|v\|^p \\
& - \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv)dx \\
& - \int_\Omega (|\nabla v|^{p-2}\nabla v \cdot \nabla u + |v|^{p-2}vu)dx.
\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv)dx \ &\leq\ \left(\int_\Omega |\nabla u|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v|^p\right)^{\frac{1}{p}} \\
&+\ \left(\int_\Omega |u|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega |v|^p\right)^{\frac{1}{p}}. \qquad (2.9)
\end{aligned}$$

Applying the inequality

$$(a+b)^\alpha (c+d)^{1-\alpha} \geq a^\alpha c^{1-\alpha} + b^\alpha d^{1-\alpha},$$

which holds for any $\alpha \in (0,1)$ and for any $a > 0, b > 0, c > 0, d > 0$, with

$$a = \int_\Omega |\nabla u|^p dx,\ \ b = \int_\Omega |u|^p dx,$$

$$c = \int_\Omega |\nabla v|^p dx,\ \ d = \int_\Omega |v|^p dx,\ \ \alpha = \frac{p-1}{p},$$

we conclude that

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv)dx \leq \|u\|^{p-1}\|v\|.$$

Hence,

$$\int_\Omega (|\nabla v|^{p-2}\nabla v \cdot \nabla u + |v|^{p-2}vu)dx \leq \|v\|^{p-1}\|u\|.$$

Therefore,

$$\begin{aligned}\langle Bu - Bv, u - v\rangle &\geq \|u\|^p + \|v\|^p - \|u\|^{p-1}\|v\| - \|v\|^{p-1}\|u\| \\ &\geq (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) \\ &\geq 0.\end{aligned}$$

Now let $u,\ v$ be such that $\langle Bu - Bv, u - v\rangle = 0$. Then we have

$$\langle Bu - Bv, u - v\rangle = (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) = 0.$$

It follows that $\|u\| = \|v\|$ and that the equality holds in (2.9). As equality in Hölder's inequality is characterized, we obtain from (2.9) that $u = kv$ a.e. in Ω, for some constant $k \geq 0$. Therefore, $k = 1$ and $u = v$ a.e. in Ω. □

Proposition 2.4. *Let G be defined in* (2.5) *then G' satisfies (H3).*

Proof. As $B = G'/p$, it suffices to show this for B. Using Sobolev's embedding theorem, Hölder's inequality and following the arguments used in the proof of proposition 2.2 one can easily see that B is continuous and bounded. It remains to be shown that B satisfies condition (S_0). That means if $\{u_n\}$ is a sequence in X such that

$$u_n \rightharpoonup u,\ Bu_n \rightharpoonup v,\ \text{and}\ \langle Bu_n, u_n\rangle \to \langle v, u\rangle$$

for some $v \in X^*$ and $u \in X$, then it follows that $u_n \to u$ in X.

By Sobolev's compact embedding theorem we have $u_n \to u$ in $L^p(\Omega)$. Since X is a reflexive Banach space, by the Lindenstrauss-Asplund-Troyanski theorem (see [30]) one can find an equivalent norm such that X with this norm is locally uniformly convex. In such a space weak convergence and norm convergence imply (strong) convergence. Thus to show $u_n \to u$ in X, we only need to show $\|u_n\| \to \|u\|$.

To this end, we first observe that

$$\lim_{n\to\infty} \langle Bu_n - Bu, u_n - u\rangle = \lim_{n\to\infty} (\langle Bu_n, u_n\rangle - \langle Bu_n, u\rangle - \langle Bu, u_n - u\rangle) = 0.$$

On the other hand, it follows from lemma 2.3 that

$$\langle Bu_n - Bu, u_n - u\rangle \geq (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|).$$

Hence $\|u_n\| \to \|u\|$ as $n \to \infty$. And therefore B satisfies condition (S_0). □

We now can apply theorem 2.1 to conclude the following.

Theorem 2.5 (Existence of L-S sequence). *Let X be a closed subspace of $W^{1,p}(\Omega)$ such that $W_0^{1,p}(\Omega) \subseteq X$ and let F, G be the two functionals defined in (2.4), (2.5). Then there exists a nonincreasing sequence of nonnegative eigenvalues $\{\mu_n\}$ obtained from the Ljusternik-Schnirelman principle such that $\mu_n \to 0$ as $n \to \infty$, where*

$$\mu_n = \sup_{H \in \mathbb{A}_n} \inf_{u \in H} F(u), \tag{2.10}$$

and each μ_n is an eigenvalue of $F'(u) = \mu G'(u)$.

Proof. The existence of such a sequence $\{\mu_n\}$ follows from theorem 2.1-(5). To verify (2.10) we observe, using (2.4), (2.5), (2.6) and (2.7), that

$$\mu_n = \mu_n G(u_n) = \mu_n \langle Bu_n, u_n\rangle = \langle Au_n, u_n\rangle = F(u_n) = a_n.$$

Combining this with (2.2) we obtain (2.10). □

3. Eigenvalue problems for the p-Laplacian

We shall establish the existence of a sequence of eigenvalues using the principle given in the previous section. We first notice that by choosing the function spaces appropriately, the Dirichlet problem $D(\Omega)$, the No-flux problem $P(\Omega)$, and the Neumann problem $N(\Omega)$ yield the same formula for weak solutions.

3.1. **Weak solutions.**

Definition 3.1.

(i) *Let X be either $W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)\oplus\mathbb{R}$ or $W^{1,p}(\Omega)$. Then a pair $(u,\lambda)\in X\times\mathbb{R}$ is a weak solution of $D(\Omega), P(\Omega), N(\Omega)$, respectively, provided that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx = \lambda\int_\Omega |u|^{p-2}uv dx,\ \textit{for any } v\in X. \tag{3.1}$$

(ii) *A pair $(u,\lambda)\in W^{1,p}(\Omega)\times\mathbb{R}$ is a weak solution of the Robin problem $R(\Omega)$ provided that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx + \int_{\partial\Omega}\beta|u|^{p-2}uv ds = \lambda\int_\Omega |u|^{p-2}uv dx, \tag{3.2}$$

for any $v\in W^{1,p}(\Omega)$.

(iii) *A pair $(u,\lambda)\in W^{1,p}(\Omega)\times\mathbb{R}$ is a weak solution of the Steklov problem $S(\Omega)$ provided that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx + \int_\Omega |u|^{p-2}uv dx = \lambda\int_{\partial\Omega} |u|^{p-2}uv ds, \tag{3.3}$$

for any $v\in W^{1,p}(\Omega)$.

In all cases, such a pair (u,λ), with u nontrivial, is called an eigenpair, λ is an eigenvalue and u is called an eigenfunction.

It follows from (3.1), (3.2) and (3.3) that all eigenvalues λ are nonnegative (by choosing $v=u$).

It will be shown that if $\partial\Omega$ is of class $C^{1,\gamma}$, then eigenfunctions of (3.1), (3.2), (3.3) belong to $C^{1,\alpha}(\bar\Omega)$. Hence ∇u exists on $\partial\Omega$ and the boundary conditions of the problems $P(\Omega)$, $N(\Omega)$, $R(\Omega)$, and $S(\Omega)$ make sense. The following lemma assures that if an eigenfunction u is smooth enough, then u solves the corresponding partial differential equation.

Lemma 3.2. *Let (u,λ) be an eigenpair, i.e., a weak solution, of (3.1) (with $X = W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $W^{1,p}(\Omega)$), or (3.2), or (3.3) such that u is in $W^{2,p}(\Omega)$, then (u,λ) solves $D(\Omega)$, $P(\Omega)$, $N(\Omega)$, $R(\Omega)$, and $S(\Omega)$, respectively.*

Proof. Let us verify this for the No-flux problem $P(\Omega)$ and the Steklov problem $S(\Omega)$. The verification for the others proceeds in a similar way.

Let $X = W_0^{1,p}(\Omega) \oplus \mathbb{R}$ and $(u,\lambda) \in W_0^{1,p}(\Omega) \oplus \mathbb{R} \times \mathbb{R}^+$ be an eigenpair of (3.1) with $u \in W^{2,p}(\Omega)$. Since u =constant on $\partial\Omega$, to show (u,λ) solves $P(\Omega)$ it remains to show

$$\int_\Omega (-\Delta_p u) v dx = \lambda \int_\Omega |u|^{p-2} u v dx, \ \forall v \in C_0^\infty(\Omega), \tag{3.4}$$

and

$$\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} ds = 0. \tag{3.5}$$

We recall the first formula of Green (see [28])

$$\int_\Omega (\Delta_p u) v dx + \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} v ds,$$

which holds for a C^1 boundary $\partial\Omega$ and for any $u \in W^{2,p}(\Omega)$, $v \in W^{1,p}(\Omega)$.

Applying Green's first identity to (3.1), we obtain

$$\int_\Omega (-\Delta_p u) v dx + \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} v ds = \lambda \int_\Omega |u|^{p-2} u v dx, \ \forall v \in W_0^{1,p}(\Omega) \oplus \mathbb{R}.$$

Thus (3.4) follows immediately, i.e., $-\Delta_p u = \lambda |u|^{p-2} u$ in Ω. Consequently,

$$\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} v ds = 0, \ \forall v \in W_0^{1,p}(\Omega) \oplus \mathbb{R}.$$

We obtain (3.5), since v =constant on $\partial\Omega$.

Now let $(u,\lambda) \in W^{2,p}(\Omega) \times \mathbb{R}^+$ be an eigenpair of (3.3). Using again Green's first identity, it follows from (3.3) that

$$\int_\Omega (-\Delta_p u) v dx + \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} v ds + \int_\Omega |u|^{p-2} u v dx = \lambda \int_{\partial\Omega} |u|^{p-2} u v ds,$$

for any $v \in W^{1,p}(\Omega)$. Thus, taking any v in $C_0^\infty(\Omega)$ we have

$$\int_\Omega (-\Delta_p u + |u|^{p-2} u) v dx = 0,$$

which implies $\Delta_p u = |u|^{p-2} u$ in Ω. Furthermore, since the range of the trace mapping $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is dense in $L^p(\partial\Omega)$, we have

$$\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} v ds = \lambda \int_{\partial\Omega} |u|^{p-2} u v ds, \forall v \in L^p(\partial\Omega).$$

Therefore, $|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda |u|^{p-2} u$ on $\partial\Omega$. □

3.2. **Existence results.** Let F and G be defined in (2.4) and (2.5). We will show that by choosing an appropriate subspace X of $W^{1,p}(\Omega)$ and appropriate functions of $a,\ b,\ \beta$ we can apply theorem 2.5 to the Dirichlet, the No-flux, the Neumann, the Robin, and the Steklov problems.

Theorem 3.3 (Existence of L-S sequences for $D(\Omega),\ P(\Omega),\ N(\Omega)$)**.** *Let F and G be defined in (2.4) and (2.5) with $a \equiv 1,\ b \equiv 0$ and $\beta \equiv 0$. Let X be $W_0^{1,p}(\Omega),\ W_0^{1,p}(\Omega) \oplus \mathbb{R}$, or $W^{1,p}(\Omega)$, then there exists a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_n^X\}$ of (3.1) obtained by using the Ljusternik-Schnirelman principle such that $\lambda_n^X = \frac{1}{\mu_n^X} - 1 \to \infty$ as $n \to \infty$, where each μ_n^X is an eigenvalue of the corresponding equation $F'(u) = \mu G'(u)$ defined in (2.10).*

Proof. With $a \equiv 1,\ b \equiv 0$ and $\beta \equiv 0$, F and G become

$$\begin{aligned} F(u) &= \int_\Omega |u|^p dx, \\ G(u) &= \int_\Omega (|\nabla u|^p + |u|^p) dx. \end{aligned}$$

And thus $F'(u) = \mu G'(u)$ is equivalent to

$$\int_\Omega |u|^{p-2} uv dx = \mu \int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla v + |u|^{p-2} uv) dx,\ \forall v \in X;$$

or

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \left(\frac{1}{\mu} - 1\right) \int_\Omega |u|^{p-2} uv dx,\ \forall v \in X.$$

Comparing the last equation to (3.1) and applying theorem 2.5 we obtain the result. □

As mentioned above, if $X = W_0^{1,p}(\Omega),\ W_0^{1,p}(\Omega) \oplus \mathbb{R}$, or $W^{1,p}(\Omega)$ we have $\{\lambda_n^D\}$, $\{\lambda_n^P\}$, and $\{\lambda_n^N\}$ are the corresponding L-S sequences of eigenvalues of $D(\Omega)$, $P(\Omega)$, and $N(\Omega)$, respectively.

Since $S_{W_0^{1,p}(\Omega)} \subset S_{W_0^{1,p}(\Omega) \oplus \mathbb{R}} \subset S_{W^{1,p}(\Omega)}$ where $S_X = \{u \in X : G(u) = 1\}$, it follows from (2.10) that $\mu_n^D \leq \mu_n^P \leq \mu_n^N$. Thus $\lambda_n^D \geq \lambda_n^P \geq \lambda_n^N$, for all n. This proves inequality (1.1) mentioned in section 1.

Theorem 3.4 (Existence of L-S sequences for $R(\Omega)$)**.** *Let X be $W^{1,p}(\Omega)$ and F, G be defined in (2.4), (2.5) with $a(x) \equiv 1$, $b(x) \equiv 0$ and $\beta(x) \equiv \beta > 0$. Then there exists a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_n\}$ of (3.2) obtained by using the Ljusternik-Schnirelman principle such that $\lambda_n = \frac{1}{\mu_n} - 1 \to \infty$ as $n \to \infty$, where each μ_n is an eigenvalue of the corresponding equation $F'(u) = \mu G'(u)$ defined in (2.10).*

Proof. With $a(x) \equiv 1,\ b(x) \equiv 0$ and $\beta(x) \equiv \beta$, F and G become

$$\begin{aligned} F(u) &= \int_\Omega |u|^p dx, \\ G(u) &= \int_\Omega (|\nabla u|^p + |u|^p) dx + \beta \int_{\partial\Omega} |u|^p ds. \end{aligned}$$

Thus $F'(u) = \mu G'(u)$ is equivalent to

$$\int_\Omega |u|^{p-2} uv dx = \mu \left(\int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla v + |u|^{p-2} uv) dx + \beta \int_{\partial\Omega} |u|^{p-2} uv ds \right),$$

for any $v \in W^{1,p}(\Omega)$; or

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \beta \int_{\partial\Omega} |u|^{p-2} uv ds = \left(\frac{1}{\mu} - 1 \right) \int_\Omega |u|^{p-2} uv dx,$$

for any $v \in W^{1,p}(\Omega)$.

Comparing the last equation to (3.2) and applying theorem 2.5 we obtain the result. □

Theorem 3.5 (Existence of L-S sequences for $S(\Omega)$). *Let X be $W^{1,p}(\Omega)$ and F, G be defined in (2.4), (2.5) with $a \equiv 0,\ b \equiv 1$ and $\beta \equiv 0$. Then there exists a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_n\}$ of (3.3) obtained using the Ljusternik-Schnirelman principle such that $\lambda_n = \frac{1}{\mu_n} \to \infty$ as $n \to \infty$, where each μ_n is an eigenvalue of the corresponding equation $F'(u) = \mu G'(u)$ defined in (2.10).*

Proof. With $a \equiv 0,\ b \equiv 1$ and $\beta \equiv 0$, F and G become

$$\begin{aligned} F(u) &= \int_{\partial\Omega} |u|^p ds, \\ G(u) &= \int_\Omega (|\nabla u|^p + |u|^p) dx. \end{aligned}$$

And thus $F'(u) = \mu G'(u)$ is equivalent to

$$\int_{\partial\Omega} |u|^{p-2} uv ds = \mu \int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla v + |u|^{p-2} uv) dx,\ \forall v \in W^{1,p}(\Omega);$$

or

$$\int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla v + |u|^{p-2} uv) dx = \frac{1}{\mu} \int_{\partial\Omega} |u|^{p-2} uv ds,\ \forall v \in W^{1,p}(\Omega).$$

Comparing the last equation to (3.3) and applying theorem 2.5 we obtain the result. □

3.3. **Remarks.**

- We notice that theorem 2.5 assures the existence of an L-S sequence of eigenvalues for any closed subspace X of $W^{1,p}(\Omega)$ and any functionals F, G defined in (2.4), (2.5). It follows that we can study eigenvalue problems with mixed boundary conditions, i.e., any combination of the Dirichlet, the No-flux, the Neumann, the Robin conditions. To give an example, let us consider the following Dirichlet-No-flux-Neumann problem:

$$DPN(\Omega):\begin{cases} -\Delta_p u &= \lambda |u|^{p-2}u, \text{ in } \Omega,\\ u &= 0, \text{ on } \Gamma_1,\\ u &= \text{constant, on } \Gamma_2,\\ \int_{\Gamma_2} |\nabla u|^{p-2}\frac{\partial u}{\partial n}ds &= 0,\\ \frac{\partial u}{\partial n} &= 0, \text{ on } \Gamma_3,\end{cases}$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open connected subsets of $\partial\Omega$ such that $\overline{\Gamma_1}\cup\overline{\Gamma_2}\cup\overline{\Gamma_3} = \Omega$.

Let $X := \{u \in W^{1,p}(\Omega) : u|_{\Gamma_1} = 0, u|_{\Gamma_2} = \text{constant}\}$. Then X is a closed subspace of $W^{1,p}(\Omega)$. We say a pair $(u, \lambda) \in X \times \mathbb{R}$ is a weak solution of (3.6) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega |u|^{p-2}uv dx, \ \forall v \in X. \tag{3.6}$$

Then we apply theorem 2.5 to obtain a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_k^{DPN}\}$ of (3.6) such that $\lambda_k^N \le \lambda_k^{DPN} \le \lambda_k^D$ for each k, where λ_k^D and λ_k^N are the k-th L-S eigenvalues of the Dirichlet problem and the Neumann problem, respectively. Furthermore, if an eigenfunction u of (3.6) is in $W^{2,p}(\Omega)$, then u solves (3.6). To see this, we use Green's first identity and follow the arguments used in the proof of lemma 3.2.

- In the next section, we will show that eigenfunctions are in $C^{1,\alpha}(\bar\Omega)$ provided that $\partial\Omega$ is regular enough. However, in order to have the result stated in lemma 3.2 we require that eigenfunctions are in $W^{2,p}(\Omega)$. When $N = 2$, the authors in [19] showed that solutions to $\Delta_p u = 0$ in Ω, $p \ne 2$, in general, do not have any better regularity than $C^{1,\alpha}$. To our knowledge, higher degrees of regularity of eigenfunctions are unknown.

4. Regularity results on eigenfunctions

In this section we shall prove boundedness of eigenfunctions and use this fact to obtain $C^{1,\alpha}(\Omega)$ and $C^{1,\alpha}(\bar\Omega)$ smoothness of (weak) eigenfunctions of the nonlinear eigenvalue problems $D(\Omega)$, $P(\Omega)$, $N(\Omega)$, $R(\Omega)$, and $S(\Omega)$.

4.1. **Boundedness for eigenfunctions.** Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary and $1 < p < \infty$. To obtain the regularity of eigenfunctions in Ω and on the boundary $\partial\Omega$ we need to show that such eigenfunctions are in $L^\infty(\Omega)$. If $p > N$ the answer follows from Sobolev's embedding theorem (see theorem A.5).

The following theorems extend lemma 3.2 of Drábek - Kufner - Nicolosi [13], which asserts that any nonnegative eigenfunction of the Dirichlet problem (1.1) is in $L^\infty(\Omega)$.

Theorem 4.1 (Boundedness for solutions of $D(\Omega), P(\Omega), N(\Omega)$). *Let X be $W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)\oplus\mathbb{R}$ or $W^{1,p}(\Omega)$ and let $(u,\lambda)\in X\times\mathbb{R}^+$ be an eigensolution of the weak form (3.1). Then $u\in L^\infty(\Omega)$.*

Proof. By Sobolev's embedding theorem it suffices to consider the case $p\le N$. In this proof, we use the Moser iteration technique (see for example [13]). Let us assume first that $u\ge 0$. For $M>0$ define $v_M(x)=\min\{u(x),M\}$.

Letting $f(x)=x$, if $x\le M$ and $f(x)=M$, if $x>M$, it follows from theorem B.3 that $v_M\in X\cap L^\infty(\Omega)$.

For $k>0$ define $\varphi=v_M^{kp+1}$, then $\nabla\varphi=(kp+1)\nabla v_M^{kp}$. It follows that $\varphi\in X\cap L^\infty(\Omega)$. Using φ as a test function in (3.1), one obtains

$$(kp+1)\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla v_M v_M^{kp}dx=\lambda\int_\Omega|u|^{p-2}uv_M^{kp+1}dx\le\lambda\int_\Omega|u|^{(k+1)p}dx,$$

or

$$\frac{(kp+1)}{(k+1)^p}\int_\Omega|\nabla v_M^{k+1}|^pdx=\lambda\int_\Omega u^{(k+1)p}dx,$$

and then

$$\frac{(kp+1)}{(k+1)^p}\int_\Omega(|\nabla v_M^{k+1}|^p+|v_M^{k+1}|^p)dx\le\left(\lambda+\frac{(kp+1)}{(k+1)^p}\right)\int_\Omega u^{(k+1)p}dx;$$

thus

$$\|v_M^{k+1}\|^p\le\left(\lambda\frac{(k+1)^p}{(kp+1)}+1\right)\|u\|_{(k+1)p}^{(k+1)p}.$$

However, by Sobolev's embedding theorem, there is a constant $c_1>0$ such that

$$\|v_M^{k+1}\|_{p^*}\le c_1\|v_M^{k+1}\|;$$

here we take $p^*=\frac{Np}{N-p}$, if $p<N$ and $p^*=2p$, if $p=N$. Thus

$$\begin{aligned}\|v_M\|_{(k+1)p^*}&\le\|v_M^{k+1}\|_{p^*}^{1/(k+1)}\\&\le c_1^{\frac{1}{k+1}}\left(\lambda\frac{(k+1)^p}{(kp+1)}+1\right)^{\frac{1}{p(k+1)}}\|u\|_{(k+1)p}.\end{aligned}$$

Using calculus, we can find a constant $c_2>0$ such that

$$\left(\lambda\frac{(k+1)^p}{(kp+1)}+1\right)^{\frac{1}{p\sqrt{k+1}}}\le c_2,$$

for any $k>0$.
Thus

$$\|v_M\|_{(k+1)p^*}\le c_1^{\frac{1}{k+1}}c_2^{\frac{1}{\sqrt{k+1}}}\|u\|_{(k+1)p}.$$

Letting $M\to\infty$, Fatou's lemma implies

$$\|u\|_{(k+1)p^*}\le c_1^{\frac{1}{k+1}}c_2^{\frac{1}{\sqrt{k+1}}}\|u\|_{(k+1)p}. \tag{4.1}$$

Choosing k_1 such that $(k_1+1)p=p^*$, then (4.1) becomes

$$\|u\|_{(k_1+1)p^*}\le c_1^{\frac{1}{k_1+1}}c_2^{\frac{1}{\sqrt{k_1+1}}}\|u\|_{p^*}.$$

Next, we choose k_2 such that $(k_2+1)p = (k_1+1)p^*$, then taking $k_2 = k$ in (4.1), we have

$$\|u\|_{(k_2+1)p^*} \le c_1^{\frac{1}{k_2+1}} c_2^{\frac{1}{\sqrt{k_2+1}}} \|u\|_{(k_2+1)p} = c_1^{\frac{1}{k_2+1}} c_2^{\frac{1}{\sqrt{k_2+1}}} \|u\|_{(k_1+1)p^*}.$$

By induction we obtain

$$\|u\|_{(k_n+1)p^*} \le c_1^{\frac{1}{k_n+1}} c_2^{\frac{1}{\sqrt{k_n+1}}} \|u\|_{(k_{n-1}+1)p^*},$$

where the sequence $\{k_n\}$ is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$, $k_0 = 0$. It is easy to see that $k_n + 1 = \left(\frac{p^*}{p}\right)^n$. Hence

$$\|u\|_{(k_n+1)p^*} \le c_1^{\sum_{i=1}^n \frac{1}{k_i+1}} c_2^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} \|u\|_{p^*}.$$

As $\frac{p}{p^*} < 1$, there is $C > 0$ such that for any $n = 1, 2, ...$

$$\|u\|_{(k_n+1)p^*} \le C\|u\|_{p^*},$$

where $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$.

We will indirectly show that $u \in L^\infty(\Omega)$. Suppose $u \notin L^\infty(\Omega)$, then there exists $\varepsilon > 0$ and a set A of positive measure in Ω such that $|u(x)| > C\|u\|_{p^*} + \varepsilon = K$, for all $x \in A$. Then

$$\liminf_{n\to\infty} \|u\|_{r_n} \ge \liminf_{n\to\infty} \left(\int_A K^{r_n}\right)^{1/r_n} = \liminf_{n\to\infty} K|A|^{1/r_n} = K > C\|u\|_{p^*},$$

which contradicts what has been established above.

If u (as an eigenfunction of (3.1)) changes sign, we consider u^+. By lemma B.2, $u^+ \in X$. Define for each $M > 0$, $v_M(x) = \min\{u^+(x), M\}$. Taking again $\varphi = v_M^{kp+1}$ as a test function in X, we obtain

$$(kp+1)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v_M v_M^{kp} dx = \lambda \int_\Omega |u|^{p-2} u v_M^{kp+1} dx,$$

which implies

$$(kp+1)\int_\Omega |\nabla u^+|^{p-2}\nabla u^+ \cdot \nabla v_M v_M^{kp} dx = \lambda \int_\Omega |u^+|^{p-2} u^+ v_M^{kp+1} dx.$$

Proceeding the same way as above we conclude that $u^+ \in L^\infty(\Omega)$. Similarly we have $u^- \in L^\infty(\Omega)$. Therefore $u = u^+ + u^-$ is in $L^\infty(\Omega)$. □

Corollary 4.2 (Global boundedness for $R(\Omega)$ solutions). *Let (u, λ) be an eigensolution of the weak form (3.2). Then $u \in L^\infty(\Omega)$.*

Proof. Let u be an eigenfunction of (3.2). We assume first that $u \ge 0$. Let $v_M = \min\{u, M\}$ and $\varphi = v_M^{kp+1}$. Theorem B.3 implies that v_M, $\varphi \in$

$W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $v_M|_{\partial\Omega} = \min\{u|_{\partial\Omega}, M\}$. Since $\beta > 0$, we have

$$\begin{aligned}(kp+1)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v_M v_M^{kp}dx &\le \\ (kp+1)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v_M v_M^{kp}dx &+ \beta\int_{\partial\Omega}|u|^{p-2}uv_M^{kp+1}ds \\ &= \lambda\int_\Omega |u|^{p-2}uv_M^{kp+1}dx.\end{aligned}$$

Then we use the argument used in the proof of theorem 4.1 to conclude that $u\in L^\infty(\Omega)$.

If u changes sign, one can easily show that both u^+ and u^- are in $L^\infty(\Omega)$. Therefore $u\in L^\infty(\Omega)$. □

Theorem 4.3 (Global boundedness for $S(\Omega)$ solutions)**.** *Let (u,λ) be an eigensolution of the weak form (3.3). Then $u\in L^\infty(\Omega)$.*

Proof. Arguing as in theorem 4.1, we can assume that $1 < p \le N$ and $u \ge 0$.

For $M > 0$ define $v_M(x) = \min\{u(x), M\}$. For $k > 0$ define $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1)\nabla v_M v_M^{kp}$. It follows that $\varphi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. Using φ as a test function we have

$$(kp+1)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v_M v_M^{kp}dx + \int_\Omega |u|^{p-2}uv_M^{kp+1}dx = \lambda\int_{\partial\Omega}|u|^{p-2}uv_M^{kp+1}ds,$$

which implies

$$\frac{(kp+1)}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx + \int_\Omega |u|^{p-2}uv_M^{kp+1}dx \le \lambda\int_{\partial\Omega}u^{(k+1)p}ds.$$

Letting $M\to\infty$, using Fatou's lemma we obtain

$$\frac{(kp+1)}{(k+1)^p}\int_\Omega |\nabla u^{k+1}|^p dx + \int_\Omega |u|^{(k+1)p}dx \le \lambda\int_{\partial\Omega}u^{(k+1)p}ds.$$

Since $\frac{(kp+1)}{(k+1)^p} < 1$ for any $k > 0$, we conclude that

$$\frac{(kp+1)}{(k+1)^p}\int_\Omega (|\nabla u^{k+1}|^p + |u^{k+1}|^p)dx \le \lambda\int_{\partial\Omega}u^{(k+1)p}dx.$$

By Sobolev's embedding theorem, there exists $c_1 > 0$ such that

$$\|u^{k+1}\|_{L^q(\partial\Omega)} \le c_1\|u^{k+1}\|,$$

here we take $q = \frac{(N-1)p}{N-p}$, if $p < N$ and $q = 2p$ if $p = N$. Thus

$$\|u\|_{L^{(k+1)q}(\partial\Omega)} \le c_1^{\frac{1}{k+1}}\left(\lambda\frac{(k+1)^p}{(kp+1)}\right)^{\frac{1}{p(k+1)}}\|u\|_{L^{(k+1)p}(\partial\Omega)}.$$

Using the iteration method in the proof of theorem 4.1 we obtain that $u\in L^\infty(\partial\Omega)$. Hence for any $k > 0$,

$$\int_\Omega u^{(k+1)p}dx \le \frac{(kp+1)}{(k+1)^p}\int_\Omega |\nabla u^{k+1}|^p dx + \int_\Omega u^{(k+1)p}dx \le \lambda\int_{\partial\Omega}u^{(k+1)p}dx.$$

Thus

$$\begin{aligned}\|u\|_{(k+1)p} &\le \left[\lambda \int_{\partial\Omega} u^{(k+1)p} dx\right]^{\frac{1}{(k+1)p}} \\ &= (\lambda|\partial\Omega|)^{\frac{1}{(k+1)p}} \|u\|_{L^\infty(\partial\Omega)} \\ &\le C\|u\|_{L^\infty(\partial\Omega)},\end{aligned}$$

where $|\partial\Omega| = \mu_{\partial\Omega}(\partial\Omega)$ is the boundary measure of $\partial\Omega$. Letting $k\to\infty$ we conclude that $u \in L^\infty(\Omega)$. □

4.2. **Regularity results on eigenfunctions.** Let Ω be a bounded domain in $\mathbb{R}^N$, $1 < p < \infty$. Consider the degenerate elliptic equation

$$-\Delta_p u(x) = f(x, u(x)), \quad \text{in } \Omega, \tag{4.2}$$

where $f : \Omega \times \mathbb{R}\to\mathbb{R}$ is a Carathéodory function, i.e.

- $x \mapsto f(x, u)$ is measurable on Ω for all $u \in \mathbb{R}$,
- $u \mapsto f(x, u)$ is continuous for a.e. $x \in \Omega$.

A function $u \in W^{1,p}_{loc}(\Omega)$ is called a weak solution of (4.2) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx = \int_\Omega f(x, u)\varphi dx, \ \forall\varphi \in C_0^\infty(\Omega).$$

The following result was established by DiBenedetto [12] and Tolksdorf [29].

Theorem 4.4. *Let u be a weak solution of (4.2) and let $g(x) = f(x, u(x))$, a.e. $x \in \Omega$. If g belongs to $L^q(\Omega)$ with $q > \frac{p}{p-1}N$, then u is a $C^{1,\alpha}(\Omega)$ function for some $\alpha > 0$. In particular, the result holds if $g \in L^\infty(\Omega)$.*

Combining theorem 4.1, corollary 4.2, theorem 4.3 and theorem 4.4 we obtain

Theorem 4.5. *If $u \in W^{1,p}(\Omega)$ is an eigenfunction of (3.1), (3.2) or (3.3), then u is in $C^{1,\alpha}(\Omega)$.*

Proof. We have shown in theorems 4.1, 4.3 and corollary 4.2 that any eigenfunction of (3.1), (3.2) or (3.3) is in $L^\infty(\Omega)$. If we define $g(x) = |u(x)|^{p-2}u(x)$ in Ω, then g is also in $L^\infty(\Omega)$. Therefore, it follows from theorem 4.4 that any eigenfunction of (3.1), (3.2) or (3.3) is in $C^{1,\alpha}(\Omega)$. □

We shall also need, as an important tool in our development, a Harnack type inequality due to Trudinger [31] (theorem 1.1, p.724 and corollary 1.1, p.725) given in the following theorem.

Theorem 4.6 (Harnack inequality). *Let $u \in W^{1,p}(\Omega)$ be a weak solution of (4.2). Suppose that for all $M < \infty$ and for all $(x, s) \in \Omega \times (-M, M)$ the condition*

$$|f(x, s)| \le b_1(x)|s|^{p-1} + b_2(x)$$

holds, where b_1, b_2 are nonnegative functions in $L^\infty(\Omega)$ depending only on M.

Then if $0 \le u(x) < M$ *in a cube* $K(3r) := K_{x_0}(3r) \subset \Omega$, *there exists a constant* C *such that*

$$\max_{K(r)} u(x) \le C \min_{K(r)} u(x),$$

where $K_{x_0}(r)$ *denotes a cube in* $\mathbb{R}^N$ *of edgelength* r *and center* x_0 *whose edges are parallel to the coordinate axes.*

Corollary 4.7. *If* $u \in W^{1,p}(\Omega)$ *is a nonnegative eigenfunction of (3.1), (3.2) or (3.3), then* u *is strictly positive in the whole domain* Ω.

Proof. Let u be a nonnegative eigenfunction, then u is in $C^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ and $u \not\equiv 0$. Suppose $u(x_0) = 0$ for some $x_0 \in \Omega$. By theorem 4.6 u is identically zero on any cube in Ω containing x_0 and thus by connectedness we obtain $u \equiv 0$ in Ω, which is a contradiction. Therefore u is strictly positive in Ω. □

Having proved that any (weak) eigenfunction of either $D(\Omega)$, $P(\Omega)$, $N(\Omega)$, $R(\Omega)$ or $S(\Omega)$ is in $L^\infty(\Omega)$, we now can use boundary regularity results for solutions of degenerate elliptic equations in Lieberman [22] to obtain that u is in $C^{1,\alpha}(\bar{\Omega})$. We state the results as follows:

Theorem 4.8. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *with* $C^{1,\gamma}$ *boundary,* $0 < \gamma \le 1$. *Let* u *be a bounded weak solution of the problem*

$$\begin{cases} -\Delta_p u(x) = g(x), & a.e. \text{ in } \Omega, \\ u = \phi, & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

with $\|u\|_\infty \le M$. *If* g *is in* $L^\infty(\Omega)$ *and* ϕ *is in* $C^{1,\gamma}(\partial\Omega)$ *with* $\|g\|_\infty \le K$ *and* $\|\phi\|_{C^{1,\gamma}(\partial\Omega)} \le L$, *then there exists a positive constant* $\alpha = \alpha(\gamma, N, p, M, K)$ *such that* u *is in* $C^{1,\alpha}(\bar{\Omega})$ *and*

$$\|u\|_{C^{1,\alpha}(\bar{\Omega})} \le C(\gamma, N, p, M, K, L, \Omega).$$

Theorem 4.9. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *with* $C^{1,\gamma}$ *boundary,* $0 < \gamma \le 1$. *Let* u *be a bounded weak solution of the problem*

$$\begin{cases} -\Delta_p u(x) = g(x), & a.e. \text{ in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = \phi(x, u), & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

with $\|u\|_\infty \le M$. *If* g *is in* $L^\infty(\Omega)$ *with* $\|g\|_\infty \le K$ *and* ϕ *satisfies the condition*

$$|\phi(x,z) - \phi(y,w)| \le L[|x-y|^\gamma + |z-w|^\gamma], \ |\phi(x,z)| \le L,$$

for all (x,z) *and* (y,w) *in* $\partial\Omega \times [-M, M]$. *Then there exists a positive constant* $\alpha = \alpha(\gamma, N, p, M, K)$ *such that* u *is in* $C^{1,\alpha}(\bar{\Omega})$ *and*

$$\|u\|_{C^{1,\alpha}(\bar{\Omega})} \le C(\gamma, N, p, M, K, L, \Omega).$$

We recall that a weak solution u in $W^{1,p}(\Omega)$ of (4.3) satisfies

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx &= \int_\Omega g \varphi dx, \ \forall \varphi \in W_0^{1,p}(\Omega), \\ u - \phi &\in W_0^{1,p}(\Omega), \end{aligned}$$

while a weak solution u in $W^{1,p}(\Omega)$ of (4.4) satisfies

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx = \int_{\Omega} g\varphi dx + \int_{\partial\Omega} \phi(x,u)\varphi dx, \ \forall\varphi \in W^{1,p}(\Omega).$$

We observe that if $\phi \equiv 0$ or $\phi(x,z) = |z|^{p-2}z$ then ϕ satisfies the hypotheses of theorems 4.8 and 4.9 for any $0 < \gamma \leq \min\{p-1, 1\}$. Therefore if $\partial\Omega$ is of class $C^{1,\gamma}$, then eigenfunctions of (3.1), (3.2) or (3.3) are in $C^{1,\alpha}(\bar{\Omega})$.

5. On the spectrum of the p-Laplacian

In this section we will study the spectra of the Dirichlet, No-flux, Neumann, Robin, and Steklov problems. In fact, as mentioned in Section 1, we will show for all the above problems the following:

- The first eigenvalue λ_1 is simple and only eigenfunctions associated to λ_1 do not change sign.
- The set of eigenvalues is closed.
- The first eigenvalue λ_1 is isolated.
- The eigenvalue λ_2 is the second eigenvalue, i.e.,

$$\lambda_2 = \inf\{\lambda : \lambda \text{ is an eigenvalue and } \lambda > \lambda_1\}.$$

Here λ_1 and λ_2 are the first two eigenvalues of the L-S sequence established (using the Ljusternik-Schnirelman principle) in section 3 (theorems 3.3, 3.4, and 3.5).

In what follows we assume that Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,\gamma}$ boundary, $\gamma > 0$, and $1 < p < \infty$.

5.1. Simplicity of the first eigenvalue.

We will show that the first element λ_1 of the L-S sequence of eigenvalues is simple and only eigenfunctions associated to λ_1 do not change sign. We recall from the regularity results of Section 4 that eigenfunctions are in $C^{1,\alpha}(\Omega)$ if $\partial\Omega$ is of class C^1 (theorem 4.5) and are in $C^{1,\alpha}(\bar{\Omega})$ if $\partial\Omega$ is of class $C^{1,\gamma}$ (theorems 4.8 and 4.9).

Let us recall the abstract problem which we have discussed in Section 2. We have established in theorem 2.5 that there exists a nonincreasing sequence of nonnegative values $\{\mu_n\}$ tending to 0 as $n \to \infty$ such that $\mu_n = \sup_{H \in \mathbb{A}_n} \inf_{u \in H} F(u)$ and $\{\mu_n\}$ are eigenvalues of $F'(u) = \mu G'(u)$, where F and G are defined in (2.4), (2.5) and $\mathbb{A}_n$ is defined in (2.2).

5.1.1. *The Dirichlet, No-flux, Neumann problems.* We first notice that $\lambda_1^D > \lambda_1^P = \lambda_1^N = 0$ which agrees with inequality (1.1), a consequence of theorem 3.3. Here $\{\lambda_n^D\}$, $\{\lambda_n^P\}$, and $\{\lambda_n^N\}$ denote the corresponding L-S sequences of eigenvalues of $D(\Omega)$, $P(\Omega)$, and $N(\Omega)$, respectively. For simplicity we will write λ_1 instead of λ_1^D, λ_1^P or λ_1^N when there is no ambiguity. It is easy to see from the characterization of μ_1 in (2.10) that

$$\lambda_1 + 1 = \frac{1}{\mu_1} = \inf\{\int_{\Omega} (|\nabla u|^p + |u|^p) dx : u \in X \text{ and } \int_{\Omega} |u|^p dx = 1\}.$$

Thus

$$\lambda_1 = \inf_{u \in X \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}, \tag{5.1}$$

where X is $W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $W^{1,p}(\Omega)$. It follows immediately that λ_1 is the smallest eigenvalue.

Theorem 5.1. *Given $X = W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $W^{1,p}(\Omega)$. Then the first eigenvalue λ_1 is simple. Moreover, all first eigenfunctions do not change sign.*

Proof. If $X = W_0^{1,p}(\Omega)$, the result is due to Anane [2] and Lindqvist [23] and their technique of proof will be used again in the proof of theorem 5.4 to show the simplicity of the first eigenvalue of the Robin problem.

In case $X = W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $W^{1,p}(\Omega)$ (No-flux or Neumann problem), by choosing $v \equiv 1$ in (3.1) we have $\lambda_1 = 0$ which is the smallest eigenvalue. And all first eigenfunctions (eigenfunctions associated with $\lambda_1 = 0$) have zero gradients and thus are nonzero constant functions. Thus the eigenspace is simple.

□

Next, let us show that any eigenfunction associated to an eigenvalue $\lambda > \lambda_1$ has to change sign.

Proposition 5.2. *Let (u, λ) be an eigenpair of (3.1) with $\lambda > \lambda_1$. Then u has to change sign in Ω.*

Proof. Again when $X = W_0^{1,p}(\Omega)$ the result is proved in Anane [2] and Lindqvist [23]. Now let $X = W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $X = W^{1,p}(\Omega)$, as (u, λ) satisfies (3.1) for any $v \in X$, by choosing $v \equiv 1$ one obtains:

$$\int_\Omega |u|^{p-2} u = 0.$$

Therefore, u has to change sign. □

5.1.2. *The Robin problem.* It follows from (2.10) and theorem 3.4 that the first eigenvalue λ_1 can be characterized as

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \{ \int_\Omega |\nabla u|^p dx + \beta \int_{\partial\Omega} |u|^p ds : \int_\Omega |u|^p dx = 1 \}.$$

Lemma 5.3. *Let u be an eigenfunction associated with λ_1, then either $u > 0$ or $u < 0$ in Ω.*

Proof. We notice that if u is a first eigenfunction, so is $|u|$. By the Harnack inequality (theorem 4.6), either $|u| > 0$ in the whole domain or $|u| \equiv 0$. To see this, let assume $|u|(x_0) = 0$ for some $x_0 \in \Omega$. Then theorem 4.6 implies that $|u|$ is identically zero in a ball centered at x_0. Covering Ω by such balls we conclude that $u \equiv 0$ in Ω, which is a contradiction. Thus, $|u|$ must be positive in Ω. By the continuity of u, either u or $-u$ is positive in the whole domain. □

Theorem 5.4. *The principal eigenvalue λ_1 is simple, i.e., if u and v are two eigenfunctions associated with λ_1, then there exists c such that $u = cv$.*

Proof. By lemma 5.3 we can assume u and v are positive in Ω. In this proof we use the technique that Lindqvist [23] used to prove the simplicity of the first eigenvalue of the Dirichlet problem. Let

$$\eta = \frac{(u+\varepsilon)^p - (v+\varepsilon)^p}{(u+\varepsilon)^{p-1}} \text{ and } \theta = \frac{(v+\varepsilon)^p - (u+\varepsilon)^p}{(v+\varepsilon)^{p-1}},$$

where ε is a positive parameter. Then

$$\nabla\eta = \left\{1 + (p-1)\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^p\right\}\nabla u - p\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^{p-1}\nabla v.$$

Since u and v are bounded (corollary 4.2), $\nabla\eta$ is in $L^p(\Omega)$ and thus η is in $W^{1,p}(\Omega)$. By symmetry, the gradient of the test-function θ in the corresponding equation for v has a similar expression with u and v interchanged.

Set $u_\varepsilon = u + \varepsilon$ and $v_\varepsilon = v + \varepsilon$. Inserting these test functions into their respective equations obtained from (3.2) and adding these equations, we obtain

$$\begin{aligned}
&\lambda_1 \int_\Omega \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)dx \\
= &\int_\Omega \left[\left\{1 + (p-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^p\right\}|\nabla u_\varepsilon|^p + \left\{1 + (p-1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^p\right\}|\nabla v_\varepsilon|^p\right]dx \\
&- \int_\Omega \left[p\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{p-1}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla v_\varepsilon + p\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{p-1}|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla u_\varepsilon\right] \\
&+\beta \int_{\partial\Omega} \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)ds \\
= &\int_\Omega (u_\varepsilon^p - v_\varepsilon^p)(|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p)dx \\
&-p\int_\Omega v_\varepsilon^p |\nabla \ln u_\varepsilon|^{p-2}\nabla \ln u_\varepsilon \cdot (\nabla \ln v_\varepsilon - \nabla \ln u_\varepsilon)dx \\
&-p\int_\Omega u_\varepsilon^p |\nabla \ln v_\varepsilon|^{p-2}\nabla \ln v_\varepsilon \cdot (\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon)dx \\
&+\beta \int_{\partial\Omega} \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)ds \\
= &L_\varepsilon + \beta \int_{\partial\Omega} \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)dx.
\end{aligned}$$

Taking $x = \nabla \ln u_\varepsilon,\ y = \nabla \ln v_\varepsilon$ and vice versa, it follows from inequality (A.3) in lemma A.2 that

$$\begin{aligned} L_\varepsilon &= \int_\Omega (u_\varepsilon^p - v_\varepsilon^p)(|\nabla \ln u_\varepsilon|^p - |\nabla \ln v_\varepsilon|^p)dx \\ &\quad -p\int_\Omega v_\varepsilon^p |\nabla \ln u_\varepsilon|^{p-2}\nabla \ln u_\varepsilon \cdot (\nabla \ln v_\varepsilon - \nabla \ln u_\varepsilon)dx \\ &\quad -p\int_\Omega u_\varepsilon^p |\nabla \ln v_\varepsilon|^{p-2}\nabla \ln v_\varepsilon \cdot (\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon)dx \\ &\geq 0. \end{aligned}$$

By the Dominated Convergence Theorem, which also holds in $L^p(\partial\Omega)$, it is apparent that

$$\lim_{\varepsilon\to 0^+} \lambda_1 \int_\Omega \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)dx = 0. \tag{5.2}$$

and

$$\lim_{\varepsilon\to 0^+} \beta \int_{\partial\Omega} \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)ds = 0. \tag{5.3}$$

(We have from theorem 4.8 that u and v are in $C^{1,\alpha}(\bar\Omega)$.)

Let us consider the case $p \geq 2$. According to inequality (A.1) in lemma A.2 we have

$$\begin{aligned} 0 &\leq C(p)\int_\Omega \left(\frac{1}{v_\varepsilon^p} + \frac{1}{u_\varepsilon^p}\right)|v_\varepsilon \nabla u_e - u_\varepsilon \nabla v_\varepsilon|^p dx \\ &\leq L_\varepsilon \\ &\leq \lambda_1 \int_\Omega \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)dx - \beta\int_{\partial\Omega}\left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)ds, \end{aligned}$$

for every $\varepsilon > 0$. Recalling (5.2), (5.3), letting $\varepsilon\to 0^+$, and using Fatou's lemma we obtain

$$\lim_{\varepsilon\to 0^+} v_\varepsilon \nabla u_e - u_\varepsilon \nabla v_\varepsilon = 0 \text{ a.e. in } \Omega,$$

and thus

$$v\nabla u = u \nabla v \text{ a.e. in } \Omega.$$

We obtain immediately that $\nabla\left(\frac{u}{v}\right) = 0$, i.e., there is a constant k such that $u = kv$ a.e. in Ω. By continuity, $u = kv$ at every point in Ω. This proves the result for the case $p \geq 2$.

The case $1 < p < 2$ is very similar. Applying inequality (A.2) in lemma A.2 we obtain

$$\begin{aligned} 0 &\leq C(p)\int_\Omega (u_\varepsilon v_\varepsilon)^p(u_\varepsilon^p + v_\varepsilon^p)\frac{|v_\varepsilon \nabla u_e - u_\varepsilon \nabla v_\varepsilon|^2}{|v_\varepsilon \nabla u_e + u_\varepsilon \nabla v_\varepsilon|^{2-p}}dx \\ &\leq L_\varepsilon \\ &\leq \lambda_1 \int_\Omega \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)dx - \beta\int_{\partial\Omega}\left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)ds, \end{aligned}$$

for every $\varepsilon > 0$. Using (5.2) and (5.3), we obtain that $u = kv$ for some constant k. □

Proposition 5.5. *Let v be an eigenfunction associated to $\lambda \neq \lambda_1$. Then v changes sign in Ω.*

Proof. Suppose that v does not change sign in Ω, then by theorem 4.6 we can assume that $v > 0$ in Ω. Let u be an eigenfunction associated to λ_1. Making similar computations as in the proof of theorem 5.4 we conclude that

$$\int_\Omega \left[\lambda_1 \frac{u^{p-1}}{u_\varepsilon^{p-1}} - \lambda \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)dx - \beta \int_{\partial\Omega} \left[\frac{u^{p-1}}{u_\varepsilon^{p-1}} - \frac{v^{p-1}}{v_\varepsilon^{p-1}}\right](u_\varepsilon^p - v_\varepsilon^p)ds.$$
$$= L_\varepsilon \geq 0.$$

Letting $\varepsilon \to 0^+$ we obtain

$$(\lambda_1 - \lambda)\int_\Omega (u^p - v^p)dx \geq 0.$$

Hence if we take ku instead of u we obtain for any $k > 0$ that

$$(\lambda_1 - \lambda)\int_\Omega (k^p u^p - v^p)dx \geq 0,$$

which yields a contradiction if we choose $k^p > \int_\Omega v^p dx / \int_\Omega u^p dx$. Therefore, v changes sign in Ω. □

5.1.3. *The Steklov problem.* Arguing as for the Dirichlet and Robin problems, one sees that the first eigenvalue λ_1 of $S(\Omega)$ can be characterized as

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega (|\nabla u|^p + |u|^p)dx : \int_{\partial\Omega} |u|^p ds = 1 \right\}.$$

Lemma 5.6. *If u_1 is an eigenfunction associated with λ_1, then either $u_1 > 0$ or $u_1 < 0$ in $\bar{\Omega}$.*

Proof. We have that $|u_1|$ is also a minimizer. It follows from the Harnack inequality (theorem 4.6) and theorem 4.9 that $|u_1| > 0$ on Ω and $|u_1|$ is in $C^{1,\alpha}(\bar{\Omega})$. Thus if there is $x_0 \in \partial\Omega$ such that $u_1(x_0) = 0$, by the Hopf lemma (see [33], theorem 5) we obtain $\frac{\partial |u_1|}{\partial n}(x_0) < 0$. But the boundary condition $|\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda |u|^{p-2}u$ imposes that $\frac{\partial |u_1|}{\partial n}(x_0) = 0$. This contradiction implies that $|u_1| > 0$ in $\bar{\Omega}$, which proves the lemma. □

Theorem 5.7. *The principal eigenvalue λ_1 is simple, i.e., if u and v are two eigenfunctions associated with λ_1, then there exists a constant c such that $u = cv$.*

Proof. The proof of this theorem is due to Martínez and Rossi [24] in which they use the technique developed in [2, 23] (see theorem 5.4). However, in order to carry the arguments made in [24], it requires that u, and v are bounded eigenfunctions. For the sake of completion we include the proof here.

By lemma 5.6 we can assume u and v are positive in $\bar{\Omega}$. We take $\eta_1 = (u^p - v^p)/u^{p-1}$ and $\eta_2 = (v^p - u^p)/v^{p-1}$ as test functions to obtain

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) dx &= \lambda_1 \int_{\partial\Omega} |u|^{p-2} u \left(\frac{u^p - v^p}{u^{p-1}}\right) ds \\ &- \int_\Omega |u|^{p-2} u \left(\frac{u^p - v^p}{u^{p-1}}\right) dx.\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) dx &= \lambda_1 \int_{\partial\Omega} |v|^{p-2} v \left(\frac{v^p - u^p}{v^{p-1}}\right) ds \\ &- \int_\Omega |v|^{p-2} v \left(\frac{v^p - u^p}{v^{p-1}}\right) dx.\end{aligned}$$

Adding both equations we get
(5.4)

$$0 = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) dx + \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) dx.$$

Using the fact that

$$\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) = \nabla u - p\frac{v^{p-1}}{u^{p-1}}\nabla v + (p-1)\frac{v^p}{u^p}\nabla u,$$

the first term of (5.4) becomes

$$\int_\Omega |\nabla u|^p dx - p\int_\Omega \frac{v^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla v \cdot \nabla u dx + (p-1)\int_\Omega \frac{v^p}{u^p}|\nabla u|^p dx =$$

$$\int_\Omega |\nabla \ln u|^p u^p dx - p\int_\Omega v^p |\nabla \ln u|^{p-2}\nabla \ln u \cdot \nabla \ln v dx + (p-1)\int_\Omega |\nabla \ln u|^p v^p dx.$$

We have an analogous expression for the second term of equation (5.4) and thus (5.4) becomes

$$\begin{aligned}0 &= \int_\Omega (u^p - v^p)(|\nabla \ln u|^p - |\nabla \ln v|^p) dx \\ &- p\int_\Omega v^p |\nabla \ln u|^{p-2}\nabla \ln u \cdot (\nabla \ln v - \nabla \ln u) dx \\ &- p\int_\Omega u^p |\nabla \ln v|^{p-2}\nabla \ln v \cdot (\nabla \ln u - \nabla \ln v) dx.\end{aligned}$$

For $p \geq 2$, letting $\{x, y\}$ be $\{\nabla \ln u, \nabla \ln v\}$ and applying inequality (A.1) in lemma A.2 we obtain

$$0 \geq \int_\Omega C(p)|\nabla \ln u - \nabla \ln v|^p (u^p + v^p) dx.$$

Hence,

$$0 = |\nabla \ln u - \nabla \ln v|.$$

This implies $u = kv$. For $p < 2$ we use inequality (A.2) in lemma A.2 to obtain the same result.

$\square$

Proposition 5.8. *Let u be an eigenfunction associated to $\lambda \neq \lambda_1$ then u changes sign on $\partial\Omega$, i.e., the sets $\{x \in \partial\Omega : u(x) > 0\}$ and $\{x \in \partial\Omega : u(x) < 0\}$ have positive boundary measure.*

Proof. Suppose that u does not change sign in Ω, then we can assume that $u > 0$ in Ω due to the Harnack inequality (theorem 4.6). Let u_1 be an eigenfunction associated to λ_1. Making similar computations as in [24] we conclude that

$$(\lambda_1 - \lambda) \int_{\partial\Omega} (u_1^p - u^p) ds \geq C \int_{\Omega} |\nabla \ln u - \nabla \ln u_1|^p (u_1^p + u^p) dx.$$

Hence if we take ku instead of u we obtain for any $k > 0$ that

$$\int_{\partial\Omega} (u_1^p - k^p u^p) ds \leq 0,$$

which yields a contradiction if we choose $k^p < \int_{\partial\Omega} u_1^p ds / \int_{\partial\Omega} u^p ds$. Therefore u changes sign in Ω.

Suppose that u does not change sign on $\partial\Omega$. We then can assume $u \leq 0$ on $\partial\Omega$. Using u^+ as a test function in (3.3) we conclude

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla u^+ dx + \int_{\Omega} |u|^{p-2} u u^+ dx = 0.$$

Since u changes sign in Ω, the left hand side is strictly positive. This contradiction implies that u changes sign on $\partial\Omega$.

$\square$

5.2. **Closedness of the set of eigenvalues.** We will show that the spectra of the Dirichlet, the No-flux, the Neumann, the Robin, and the Steklov problems are closed. Precisely, we will prove that the sets of all numbers λ that satisfy (3.1), (3.2) or (3.3), respectively, are closed.

We first show the closedness of the sets of eigenvalues of the Dirichlet, the No-flux, the Neumann, and the Robin problems.

Theorem 5.9. *The sets of eigenvalues of $D(\Omega)$, $P(\Omega)$, $N(\Omega)$, and $R(\Omega)$ (equations (3.1) and (3.2)) are closed.*

Proof. Let X be either $W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $W^{1,p}(\Omega)$. Let $\{(u_n, \gamma_n)\}$ be a sequence of eigenpairs of (3.1) or (3.2) such that $\gamma_n \to \gamma$ for some $\gamma \geq 0$. Without loss of generality we can assume $\|u_n\| = 1$ and thus $\{u_n\}$ has a weakly convergent subsequence, i.e., we may assume that $u_n \rightharpoonup u$ in X. By lemma 2.3,

$$\langle B(u_n) - B(u), u_n - u \rangle \geq (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|).$$

However, as the (u_n, γ_n)'s are eigenpairs, the left hand side equals

$$\langle B(u_n) - B(u), u_n - u \rangle = (\gamma_n + 1)\langle Au_n, u_n - u \rangle + \langle Bu, u_n - u \rangle$$

which tends to 0, as $n \to \infty$.

Since $\langle B(u_n) - B(u), u_n - u\rangle \to 0$ we conclude that $\|u_n\| \to \|u\|$ as $n \to \infty$. It follows that $u_n \to u$ in X, since $W^{1,p}(\Omega)$ is locally uniformly convex (with respect to an equivalent norm, see the proof of proposition 2.4 or [30]).

To show that γ is an eigenvalue of (3.1) or (3.2) and u is an associated eigenfunction we need to show for any $v \in X$ as $n \to \infty$,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx \quad \to \quad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \tag{5.5}$$

$$\int_\Omega |u_n|^{p-2} u_n v dx \quad \to \quad \int_\Omega |u|^{p-2} u v dx, \tag{5.6}$$

$$\int_{\partial\Omega} |u_n|^{p-2} u_n v dx \quad \to \quad \int_{\partial\Omega} |u|^{p-2} u v dx. \tag{5.7}$$

Let $w_n = |\nabla u_n|^{p-2}\nabla u_n$ and $w = |\nabla u|^{p-2}\nabla u$. Then as $u_n \to u$ in $W^{1,p}(\Omega)$

$$w_n(x) \to w(x), \text{ a.e. in } \Omega \quad \text{and} \quad \int_\Omega |w_n|^{\frac{p}{p-1}} dx \to \int_\Omega |w|^{\frac{p}{p-1}} dx.$$

It follows from lemma A.1 that $w_n \to w$ in $L^{p/(p-1)}(\Omega)$. Thus, by Hölder's inequality we obtain (5.5). Similarly, we have (5.6) and (5.7). □

We now show the closedness of the spectrum of the Steklov problem.

Theorem 5.10. *The set of eigenvalues of (3.3) is closed.*

Proof. Let $\{(u_n, \gamma_n)\}$ be a sequence of eigenvalues of (3.3) such that $\gamma_n \to \gamma$ for some $\gamma \geq 0$. Without loss of generality we can assume $\|u_n\| = 1$ and thus $\{u_n\}$ has a weakly convergent subsequence, i.e., we may assume that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$.

We recall that each (u_n, γ_n) satisfies

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx + \int_\Omega |u_n|^{p-2} u_n v dx = \gamma_n \int_{\partial\Omega} |u_n|^{p-2} u_n v ds,$$

or

$$\langle B(u_n), v\rangle = \gamma_n \int_{\partial\Omega} |u_n|^{p-2} u_n v ds,$$

for all $v \in W^{1,p}(\Omega)$. By lemma 2.3, we have

$$\langle B(u_n) - B(u), u_n - u\rangle \geq (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|).$$

However, the left hand side equals

$$\langle B(u_n) - B(u), u_n - u\rangle \quad = \quad \gamma_n \int_{\partial\Omega} |u_n|^{p-2} u_n (u_n - u) dx + \langle Bu, u_n - u\rangle,$$

which tends to 0 as $n \to \infty$.

The rest of the proof follows from the argument that we used in the proof of theorem 5.9. Therefore, (u, γ) is an eigenpair of (3.3). □

5.3. **Isolation of the first eigenvalue.**

5.3.1. *The Dirichlet, No-flux, Neumann, and Robin problems.* Let us recall (see appendix C) that if u is a continuous function on Ω then the set $Z(u) = \{x \in \Omega : u(x) = 0\}$ is called the zero set of u and any component ω of $\Omega \setminus Z(u)$ is called a nodal domain of u.

Given λ, an eigenvalue of either (3.1) or (3.2), and u an associated eigenfunction to λ, we define:

$$\begin{aligned} N(u) &= \text{the number of components of } \Omega \setminus Z(u), \\ N(\lambda) &= \sup\{N(u) : u \text{ is an associated eigenfunction to } \lambda\}. \end{aligned}$$

We will show that $N(\lambda)$ is finite.

Theorem 5.11. *Let (u, λ) be an eigenpair of (3.1) or (3.2) and let ω be a nodal domain of u. Then there exist two constants c and r independent of ω, u and λ such that the Lebesgue measure*

$$|\omega| \geq [(\lambda + 1)c^p)]^r =: C > 0.$$

Therefore $N(\lambda) \leq |\Omega|/C$. (In the case $X = W_0^{1,p}(\Omega) \oplus \mathbb{R}$ we assume further that $\partial\Omega$ is connected so that theorem C.3 holds.)

Proof. Let X be either $W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $W^{1,p}(\Omega)$. We will prove the theorem for (u, λ) satisfying

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \beta |u|^{p-2} uv ds = \lambda \int_\Omega |u|^{p-2} uv dx, \ \forall v \in X, \tag{5.8}$$

with a given $\beta \geq 0$. Thus, if we take $\beta = 0$ we obtain the result for the Dirichlet, the No-flux, and the Neumann problems and if we take $\beta > 0$, $X = W^{1,p}(\Omega)$ we obtain the result for the Robin problem.

We first notice that the regularity results from section 4 assure that u is in $C(\bar{\Omega})$. Let $\bar{u} = u\chi_\omega$ be the restriction of u on ω. Then by theorem C.3 we have $\bar{u} \in X$. Furthermore, $\nabla \bar{u} = \nabla u \chi_\omega$. Taking the test function v in (5.8) to be $\bar{u}$ and using lemmas C.3 and C.4, we obtain

$$\int_\Omega |\nabla \bar{u}|^p dx + \beta \int_{\partial\Omega} |\bar{u}|^p ds = \lambda \int_\Omega |\bar{u}|^p dx = \lambda \int_\omega |u|^p dx.$$

Adding $\int_\Omega |\bar{u}|^p dx$ to both sides and using Hölder's inequality, we conclude

$$\int_\Omega (|\nabla \bar{u}|^p + |\bar{u}|^p) dx + \beta \int_{\partial\Omega} |\bar{u}|^p dx = (\lambda + 1) \int_\omega |u|^p dx.$$

Thus

$$\begin{aligned} \int_\Omega (|\nabla \bar{u}|^p + |\bar{u}|^p) dx &\leq (\lambda + 1)|\omega|^{1-\frac{p}{p^*}} (\int_\omega |u|^{p^*} dx)^{\frac{p}{p^*}} \\ &= (\lambda + 1)|\omega|^{1-\frac{p}{p^*}} (\int_\Omega |\bar{u}|^{p^*} dx)^{\frac{p}{p^*}}. \end{aligned}$$

Here we choose $p^* = \frac{Np}{N-p}$, if $p < N$ and $p^* = 2p$, if $p \geq N$.

By Sobolev's embedding theorem one has

$$\|\bar{u}\|_{L^{p^*}(\Omega)} \le c\|\bar{u}\| = c\left(\int_\Omega (|\nabla \bar{u}|^p + |\bar{u}|^p)dx\right)^{\frac{1}{p}},$$

which implies

$$\|\bar{u}\|^p_{L^{p^*}(\Omega)} \le c^p(\lambda+1)|\omega|^{1-\frac{p}{p^*}}\|\bar{u}\|^p_{L^{p^*}(\Omega)}.$$

Since $\bar{u} \neq 0$, we conclude that $|\omega| \ge ((\lambda+1)c^p)^r$, where $r = -N/p$, if $p < N$ and $r = -2$, if $p \ge N$. □

Corollary 5.12. *Let (u, λ) be an eigenpair of (3.1) or (3.2) and let $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ such that $|\Omega^+| > 0$. Then there exist two constants c and r independent of u and λ such that*

$$|\Omega^+| \ge [(\lambda+1)c^p)]^r = \beta > 0.$$

The result also holds for $\Omega^- = \{x \in \Omega : u(x) < 0\}$, if $|\Omega^-| > 0$. In fact, the corollary is still valid if $\partial\Omega$ is only of class C^1 (so that the embedding theorem A.5 can be applied) instead of class $C^{1,\gamma}$.

Proof. By lemma B.2 we have u^+ is in X (the lemma does not require any regularity on the boundary $\partial\Omega$). Replacing $\bar{u}$ in the proof of theorem 5.11 by u^+ and using the same argument, we obtain the corollary. □

We are in the position to prove the isolation of the first eigenvalue λ_1.

Theorem 5.13. *The first eigenvalue λ_1 of (3.1) or (3.2) is isolated.*

Proof. Let X be $W_0^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$ or $W^{1,p}(\Omega)$. Suppose λ_1 is not isolated. Then by theorem 5.9 there exists a sequence of eigenpairs $\{(u_n, \gamma_n)\}$ such that as $n \to \infty$, $u_n \to u$ in X and $\gamma_n \to \lambda_1$, where u is an eigenfunction corresponding to λ_1.

We can assume that $\|u_n\| = \|u\| = 1$ for any n and that $u > 0$ in Ω. Define for each n

$$\Omega_n^- = \{x \in \Omega : u_n(x) < 0\} \text{ and } \Omega_n^+ = \{x \in \Omega : u_n(x) > 0\}.$$

By corollary 5.12, there exists $a > 0$ such that $|\Omega_n^-| \ge a > 0$ for any n, i.e., the measure $|\Omega_n^-|$ is uniformly bounded from below. Since u is continuous and positive on Ω, there exists $\varepsilon > 0$ such that $|\Omega_\varepsilon| > |\Omega| - a/4$, where $\Omega_\varepsilon = \{x \in \Omega : u(x) > \varepsilon\}$. By Egoroff's theorem there is a measurable subset E of Ω_ε such that $|E| > |\Omega_\varepsilon| - a/4$ and u_n converges uniformly to u on E. Thus there exists n_ε such that $|u_n(x) - u(x)| < \varepsilon/2$, for any $x \in E$ and any $n \ge n_\varepsilon$. In particular $E \subset \Omega^+_{n_\varepsilon}$. Thus $|\Omega^-_{n_\varepsilon}| < |\Omega| - |E| < |\Omega| - |\Omega_\varepsilon| + a/4 < |\Omega| - |\Omega| + a/2 = a/2$. We have arrived at a contradiction. Therefore λ_1 is isolated. □

5.3.2. *The Steklov problem.* Given λ, an eigenvalue of (3.3) and u an eigenfunction associated to λ, theorem 4.9 implies that the eigenfunction u is in $C^{1,\alpha}(\bar{\Omega})$. Thus we may define:

$$\begin{aligned} Z(u) &= \{x \in \bar{\Omega} : u(x) = 0\}, \\ N(u) &= \text{the number of components of } \bar{\Omega} \setminus Z(u), \\ N(\lambda) &= \sup\{N(u) : u \text{ is an associated eigenfunction to } \lambda\}. \end{aligned}$$

We will show again that $N(\lambda)$ is finite.

Theorem 5.14. *Let (u, λ) be a (weak) eigenpair of $S(\Omega)$ and let ω be a component of $\bar{\Omega} \setminus Z(u)$. Then there exists a constant C independent of ω, u and λ such that*

$$|\partial\Omega \cap \omega| \geq C\lambda^{-\beta},$$

where $\beta = (N-1)/(p-1)$, if $1 < p < N$ and $\beta = 2$, if $p \geq N$. Here $|A|$ denotes the boundary measure of a measurable subset A of $\partial\Omega$. Consequently, $N(\lambda) \leq |\partial\Omega|\lambda^{\beta}/C$.

Proof. Let $\bar{u} = u\chi_\omega$, then by theorem C.3 we have $\bar{u} \in W^{1,p}(\Omega)$. Taking $\bar{u}$ as a test function in (3.3) we obtain that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \bar{u} + |u|^{p-2}u\bar{u}dx = \lambda \int_{\partial\Omega} |u|^{p-2}u\bar{u}ds.$$

Hence, using Hölder's inequality in $L^p(\partial\Omega)$ and lemma C.4 we have

$$\|\bar{u}\|^p_{W^{1,p}(\Omega)} \leq \lambda \left(\int_{\partial\Omega} |\bar{u}|^{p\alpha}ds\right)^{1/\alpha} |\partial\Omega \cap \partial\omega|^{1/\beta}.$$

If $1 < p < N$, we choose $\alpha = (N-1)/(N-p)$ and $\beta = (N-1)/(p-1)$. Then we use Sobolev's embedding theorem (theorem A.5-(iii)) to conclude that there exists a constant C such that

$$\|\bar{u}\|^p_{L^{\alpha p}(\partial\Omega)} \leq C\|\bar{u}\|^p_{W^{1,p}(\Omega)}.$$

If $p \geq N$ we choose $\alpha = \beta = 2$ and we argue as before using the embedding $W^{1,p}(\Omega) \hookrightarrow L^{\alpha p}(\partial\Omega)$. □

Corollary 5.15. *Let u be an eigenfunction associated to $\lambda \neq \lambda_1$, then there exists a constant C such that*

$$|\partial\Omega^+| \geq C\lambda^{-\beta} \textit{ and } |\partial\Omega^-| \geq C\lambda^{-\beta},$$

where $\partial\Omega^+ = \partial\Omega \cap \{u > 0\}, \partial\Omega^- = \partial\Omega \cap \{u < 0\}$.

We can now establish the isolation of λ_1.

Theorem 5.16. *The principal eigenvalue λ_1 of $S(\Omega)$ is isolated. That is, there exists $a > \lambda_1$ such that λ_1 is the unique eigenvalue in $[0, a]$.*

Proof. Suppose λ_1 is not isolated. Then by theorem 5.10, there exists a sequence of eigenpairs $\{(u_n,\gamma_n)\}$ such that as $n\to\infty$, $u_n\to u$ in $W^{1,p}(\Omega)$ and $\gamma_n\to\lambda_1$, where u is an eigenfunction associated to λ_1.

We can assume $\|u_n\| = \|u\| = 1$ for any n and $u > 0$ in $\bar{\Omega}$. Define for each n

$$\partial\Omega_n^- = \{x \in \partial\Omega : u_n(x) < 0\} \text{ and } \partial\Omega_n^+ = \{x \in \partial\Omega : u_n(x) > 0\}.$$

Then, by corollary 5.15, there is $a > 0$ such that $|\partial\Omega_n^-| \geq a > 0$ for any n, i.e., the measure $|\partial\Omega_n^-|$ is uniformly bounded from below.

Since u is continuous and positive on $\partial\Omega$ there exists $\varepsilon > 0$ such that $|\partial\Omega_\varepsilon| > |\partial\Omega| - a/4$, where $\partial\Omega_\varepsilon = \{x \in \partial\Omega : u(x) > \varepsilon\}$. By Egoroff's theorem, there is a subset E of $\partial\Omega_\varepsilon$ such that $|E| > |\partial\Omega_\varepsilon| - a/4$ and u_n converges uniformly to u on E. Thus there exists n_ε such that $|u_n(x) - u(x)| < \varepsilon/2$ for any $x \in E$ and any $n \geq n_e$. In particular $E \subset \partial\Omega_{n_\varepsilon}^+$. Thus $|\partial\Omega_{n_\varepsilon}^-| < |\partial\Omega| - |E| < |\partial\Omega| - |\partial\Omega_\varepsilon| + a/4 < |\partial\Omega| - |\partial\Omega| + a/2 = a/2$. We have arrived at a contradiction. Therefore λ_1 is isolated.

□

5.4. **On the second eigenvalue.** In this subsection we will show that the eigenvalue λ_2 of the L-S sequence of eigenvalues whose existence was established in theorems 3.3, 3.4, and 3.5 is actually the smallest eigenvalue of the spectrum that is greater than the principal eigenvalue λ_1. This work is motivated by the result in [3] in which Anane and Tsouli consider the Dirichlet problem.

We begin by proving an interesting property on the number of nodal domains of a given eigenvalue of the Dirichlet, the Neumann or the Robin problems.

Proposition 5.17. *For any eigenvalue λ of (3.1) or (3.2), we have*

$$\lambda_{N(\lambda)} \leq \lambda.$$

Here $N(\lambda)$ is the maximal number of nodal domains associated with λ (see theorem 5.11) and $\lambda_{N(\lambda)}$ is the $N(\lambda)$-th eigenvalue taken from the L-S sequence of theorem 3.3 or theorem 3.4.

Proof. Let $r = N(\lambda)$, then there is an eigenfunction $u \neq 0$ associated to λ such that $N(u) = r$. Let $\omega_1, \omega_2, ..., \omega_r$ be the r-components of $\Omega \setminus Z(u)$. For $i = 1, 2, ..., r$ we define

$$v_i(x) = \begin{cases} \dfrac{u(x)}{[\int_{\omega_i} |u|^p dx]^{1/p}}, & \text{if } x \in \overline{\omega_i}, \\ 0, & \text{if } x \in \bar{\Omega} \setminus \overline{\omega_i}. \end{cases}$$

By theorem C.3 we have that $v_i \in X$ $(=W_0^{1,p}(\Omega),\ W_0^{1,p}(\Omega) \oplus \mathbb{R}$, or $W^{1,p}(\Omega))$, for $i = 1, 2, ..., r$. Let X_r denote the subspace of X spanned by $\{v_1, v_2, ..., v_r\}$. Since the v_i's are linearly independent, we have that

$\dim X_r = r$.

For each $v \in X_r,\ v = \sum_{i=1}^r \alpha_i v_i$, we have:

$$F(v) = \int_\Omega |v|^p dx = \sum_{i=1}^r |\alpha_i|^p F(v_i) = \sum_{i=1}^r |\alpha_i|^p.$$

Thus the map $v \mapsto F(v)^{1/p}$ defines a norm on X_r. Hence the compact set S_r defined by

$$S_r = \left\{ v \in X_r : F(v) = \frac{1}{\lambda+1} \right\},$$

which can be identified with the unit sphere of $\mathbb{R}^r$, has genus r.

By choosing $v = v_i$ as a test function, we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v_i dx + \beta \int_{\partial\Omega} |u|^{p-2} u v_i ds = \lambda \int_\Omega |u|^{p-2} u v_i dx,$$

which (by lemma C.4) becomes

$$\int_{\omega_i} |\nabla v_i|^p dx + \beta \int_{\partial\Omega \cap \partial\omega_i} |v_i|^p ds = \lambda \int_{\omega_i} |v_i|^p dx,$$

or

$$G(v_i) = (\lambda+1) F(v_i), \text{ for } i = 1, 2, ..., r.$$

Thus, for $v \in S_r$, we have

$$G(v) = (\lambda+1) \sum_{i=1}^r |\alpha_i|^p F(v_i) = (\lambda+1) \sum_{i=1}^r |\alpha_i|^p = (\lambda+1) F(v) = 1.$$

This implies $S_r \subset S_G$. Hence,

$$\frac{1}{1+\lambda_r} = \mu_r = \sup_{H \in \mathbb{A}_r} \inf_{v \in H} F(v) \geq \inf_{v \in S_r} F(v) = \frac{1}{1+\lambda}.$$

Therefore $\lambda_r \leq \lambda$. □

We have an analogous result for the Steklov problem.

Proposition 5.18. *For any eigenvalue λ of (3.3), we have*

$$\lambda_{N(\lambda)} \leq \lambda.$$

Here $N(\lambda)$ is the maximal number of nodal domains associated with λ (see theorem 5.14) and $\lambda_{N(\lambda)}$ is the $N(\lambda)$-th eigenvalue taken from the L-S sequence of theorem 3.5.

Proof. Let $r = N(\lambda)$ then there is an eigenfunction $u \neq 0$ associated to λ such that $N(u) = r$. Let $\omega_1, \omega_2, ..., \omega_r$ be the r-components of $\bar{\Omega} \setminus Z(u)$. For $i = 1, 2, ..., r$ we define

$$v_i(x) = \begin{cases} \dfrac{u(x)}{[\int_{\partial\Omega \cap \omega_i} |u|^p dx]^{1/p}}, & \text{if } x \in \overline{\omega_i}, \\ 0, & \text{if } x \in \bar{\Omega} \setminus \overline{\omega_i}. \end{cases}$$

Then, by theorem C.3, we have $v_i \in W^{1,p}(\Omega)$, for $i = 1, 2, ..., r$.

Let X_r denote the subspace of $W^{1,p}(\Omega)$ spanned by $\{v_1, v_2, ..., v_r\}$. For each $v \in X_r$, $v = \sum_{i=1}^r \alpha_i v_i$, we have

$$F(v) = \int_{\partial\Omega} |u|^p dx = \sum_{i=1}^r |\alpha_i|^p F(v_i) = \sum_{i=1}^r |\alpha_i|^p.$$

Thus the map $v \mapsto F(v)^{1/p}$ is a norm on X_r. Hence the compact set S_r defined by

$$S_r = \left\{ v \in X_r : F(v) = \frac{1}{\lambda} \right\},$$

which can be identified with the unit sphere of $\mathbb{R}^r$, it has genus r.

By choosing $v = v_i$ as a test function, we obtain:

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v_i dx + \int_\Omega |u|^{p-2} u v_i dx = \lambda \int_{\partial\Omega} |u|^{p-2} u v_i ds,$$

which becomes

$$\int_{\Omega\cap\omega_i} (|\nabla v_i|^p + |v_i|^p) dx = \lambda \int_{\partial\Omega\cap\omega_i} |v_i|^p ds,$$

or

$$G(v_i) = \lambda F(v_i), \text{ for } i = 1, 2, ..., r.$$

Thus, for $v \in S_r$, we have:

$$G(v) = \lambda \sum_{i=1}^r |\alpha_i|^p F(v_i) = \lambda \sum_{i=1}^r |\alpha_i|^p = \lambda F(v) = 1.$$

This implies $S_r \subset S_G$. Hence

$$\frac{1}{\lambda_r} = \sup_{H \in \mathbb{A}_r} \inf_{v \in H} F(v) \geq \inf_{v \in S_r} F(v) = \frac{1}{\lambda}.$$

Therefore $\lambda_r \leq \lambda$. □

Theorem 5.19. *For any of the problems,*

$$\lambda_2 = \inf\{\lambda : \lambda \textit{ is an eigenvalue and } \lambda > \lambda_1\}.$$

Proof. Let $\gamma = \inf\{\lambda : \lambda \text{ is an eigenvalue and } \lambda > \lambda_1\}$. It suffices to show $\lambda_2 \leq \gamma$.

Since the set of eigenvalues is closed and λ_1 is isolated, γ is an eigenvalue different from λ_1. If v is an eigenfunction associated to γ then v changes sign. Thus $\lambda_2 \leq \lambda_{N(\gamma)}$. On the other hand, propositions 5.17 and 5.18 imply that $\lambda_{N(\gamma)} \leq \gamma$.

Therefore,

$$\lambda_2 = \gamma.$$

□

5.5. **Remarks.** We have the following remarks:

- For the Dirichlet problem, as mentioned by Lindqvist in [23] and Anane, Tsouli in [3], we do not need any regularity of $\partial\Omega$. There are three reasons. Firstly, Sobolev's embedding theorem A.5 holds for $W_0^{1,p}(\Omega)$ without assuming that $\partial\Omega$ is of class C^1. Secondly, theorem 5.13 (isolation of λ_1) uses corollary 5.12 which in turn uses theorem B.2; the latter theorem is valid for any arbitrary bounded domain Ω in $\mathbb{R}^N$. Thirdly, theorem 5.19 uses theorem 4.8 which holds in $W_0^{1,p}(\Omega)$ for any arbitrary bounded domain Ω.
- For the Neumann problem, the existence of the L-S sequence was established by Friedlander in [15] and in this case we only need C^1 boundary regularity for $\partial\Omega$ so that the compact embedding
$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$
(theorem A.5) holds.
- For the No-flux problem, similar to the Neumann problem, we only need that $\partial\Omega$ is of class C^1 which is used in Sobolev's embedding theorem A.5, to establish the existence of the L-S sequence of eigenvalues, simplicity and isolation of the first eigenvalue and closedness of the spectrum. However, to establish the characterization of the second eigenvalue λ_2 in theorem 5.19, we need Lieberman's boundary regularity of eigenfunctions (theorem 4.9) and connectedness of $\partial\Omega$ so that we can apply theorem C.3. Thus we require that $\partial\Omega$ is of class $C^{1,\gamma}$ and connected.
- The Robin problem which includes the Neumann problem (when $\beta = 0$) requires $\partial\Omega$ be of class C^1 to obtain the L-S sequence of eigenvalues. Moreover, in the case $\beta > 0$, it requires $\partial\Omega$ be of class $C^{1,\gamma}$ in order to have the simplicity and isolation of the principal eigenvalue together with the characterization of the second eigenvalue.
- For the Steklov problem, Bonder and Rossi established the existence of the L-S sequence of eigenvalues in [5] and Martínez, and Rossi showed the isolation and simplicity for the first eigenvalue in [24]. Since the eigenvalue appears as a multiplier in the boundary term, we need boundary regularity for eigenfunctions and thus we require $\partial\Omega$ be of class $C^{1,\gamma}$ for some $\gamma > 0$.
- There are still some interesting open problems that we have not answered:
  1. Are there any eigenvalues different from L-S eigenvalues?
  2. Are these spectra discrete?
  3. Are all of the eigenvalues bifurcation points for equations involving higher order perturbations?
  4. What type of Fredholm alternative may be derived for such eigenvalues? And what type of resonance results? We note that for the Dirichlet problem much is known about the above questions in the

linear case $p = 2$ (see [8, 9, 36]) and in the one-dimensional case $N = 1$ (see Nečas [25]). Also if $p \geq 2$, it is shown in [27] that any L-S eigenvalue of the Dirichlet problem is a bifurcation point.

Appendix A. Some useful results

The lemma below concerns Young functions which play an important role in Orlicz-Sobolev space theory. In this paper, the result is used in proposition 2.2, proposition 2.4, and theorem 5.9.

Lemma A.1. *Let Ω be a domain in $\mathbb{R}^N$ and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a Young function which satisfies a Δ_2-condition, i.e., there is $c > 0$ such that $\Phi(2t) \leq c\Phi(t)$ for all $t \geq 0$. If $\{u_n\}$ is a sequence of integrable functions in Ω such that*

$$u(x) = \lim_{n\to\infty} u_n(x), \;\; a.e.\; x \in \Omega \;\; and \;\; \int_\Omega \Phi(|u|)dx = \lim_{n\to\infty} \int_\Omega \Phi(|u_n|)dx,$$

then

$$\lim_{n\to\infty} \Phi(|u_n - u|)dx = 0.$$

Proof. See Rao and Ren [26] (theorem 12, page 83) for the proof. □

The following inequalities are due to Lindqvist and are used to show the simplicity of the first eigenvalue in theorem 5.4 and theorem 5.7.

Lemma A.2.

a) *Let $p \geq 2$ then for all $x, y \in \mathbb{R}^N$*

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y - x) + C(p)|x - y|^p. \tag{A.1}$$

b) *Let $1 < p < 2$, then for all $x, y \in \mathbb{R}^N$,*

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y - x) + C(p)\frac{|x - y|^2}{(|x| + |y|)^{2-p}}. \tag{A.2}$$

c) *For any $x \neq y$, $p > 1$,*

$$|y|^p > |x|^p + p|x|^{p-2}x \cdot (y - x). \tag{A.3}$$

In the above $C(p)$ is a constant depending only on p.

Proof. We refer to Linqvist [23] for the proof of this lemma. □

Let u be a nonnegative function in $W^{1,p}(\Omega)$, where Ω is an open set in $\mathbb{R}^N$. Then u can be approximated by a sequence of nonnegative functions in $C(\bar{\Omega}) \cap W^{1,p}(\Omega)$.

Lemma A.3. *Let u be in $W^{1,p}(\Omega)$, $u \geq 0$. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $W^{1,p}(\Omega)$. Then the sequence $\{u_n^+\}$ also converges to u in $W^{1,p}(\Omega)$.*

Proof. For each n, let $v_n := u_n^+$ and $\Omega_n := \{x \in \Omega : u_n(x) < 0\}$. Then $v_n := u_n \chi_{\Omega_n^c}$, where A^c denotes the complement of A in Ω.

As $|v_n - u| \leq |u_n - u|$, $v_n \to u$ in $L^p(\Omega)$. It remains to show that as $n \to \infty$,

$$\int_\Omega |\nabla v_n - \nabla u|^p dx \to 0,$$

which is equivalent to showing

$$\int_{\Omega_n} |\nabla u|^p dx \to 0.$$

By lemma B.2, $\nabla u = 0$ on the set $S := \{x \in \Omega : u(x) = 0\}$, hence it suffices to show $|\Omega_n \cap \Omega^+| \to 0$ as $n \to \infty$, where $\Omega^+ = \{x \in \Omega : u(x) > 0\}$.

To this end, we recall that u_n converges to u in measure, i.e., given $\varepsilon > 0$, there exists n_ε such that for any $n > n_\varepsilon$ we have $|\{x \in \Omega : |u_n(x) - u(x)| > \varepsilon\}| < \varepsilon$. It follows that $|\Omega_n \cap \{x \in \Omega : u(x) \geq \varepsilon\}| < \varepsilon$ for $n > n_\varepsilon$. Hence

$$\limsup_{n\to\infty} |\Omega_n \cap \Omega^+| \leq \varepsilon + \limsup_{n\to\infty} |\Omega_n \cap \{x \in \Omega : 0 < u(x) < \varepsilon\}| = O(\varepsilon).$$

Letting $\varepsilon \to 0$ we conclude that $\limsup_{n\to\infty} |\Omega_n \cap \Omega^+| = 0$. □

If a sequence $\{u_n\}$ in $C^1(\bar{\Omega})$ converges to some u in $W^{1,p}(\Omega)$, $u \geq 0$, then the sequence $\{u_n^+\}$ is in $C(\bar{\Omega}) \cap W^{1,p}(\Omega)$ and converges to u. In fact, we can construct such a sequence in $C_0^\infty(\Omega)$ if u is in $W_0^{1,p}(\Omega)$.

Corollary A.4. *Let u be in $W_0^{1,p}(\Omega)$, $u \geq 0$. Then there exists a sequence of nonnegative functions in $C_0^\infty(\Omega)$ converging to u in $W_0^{1,p}(\Omega)$.*

Proof. Let $\{u_n\}$ be a sequence in $C_0^\infty(\Omega)$ converging to u in $W_0^{1,p}(\Omega)$. It follows from the proof of lemma A.3 that the sequence $\{u_n^+\}$ also converges to u. Thus we can assume u is in $C_0(\Omega) \cap W_0^{1,p}(\Omega)$. Define for each $\varepsilon > 0$ the convolution $u_\varepsilon(x) := J_\varepsilon * u(x) = \int_\Omega J_\varepsilon(x-y)u(y)dy$ where J_ε is a mollifier. Then u_ε is nonnegative and is in $C_0^\infty(\Omega)$ if ε, is sufficiently small. Applying lemma 3.15 of [1] we conclude that $\lim_{\varepsilon\to 0} u_\varepsilon = u$ in $W^{1,p}(\Omega)$. □

Theorem A.5 (Sobolev's embedding theorem.)**.** *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary.*

- (i) *If $1 < p < N$, the space $W^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, $p^* = Np/(N-p)$, and compactly embedded in $L^q(\Omega)$ for any $1 \leq q < p^*$.*
- (ii) *If $0 < 1 - \frac{N}{p} < 1$, the space $W^{1,p}(\Omega)$ is continuously embedded in $C^\alpha(\bar{\Omega})$, $\alpha = 1 - N/p$, and compactly embedded in $C^\beta(\bar{\Omega})$ for any $\beta < \alpha$.*
- (iii) *If $1 < p < N$, then under the trace mapping the space $W^{1,p}(\Omega)$ is continuously embedded in $L^{q^*}(\partial\Omega)$, $q^* = (Np-p)/(N-p)$, and compactly embedded in $L^q(\partial\Omega)$ for any $1 \leq q < q^*$.*

For the proof we refer to Adams [1], Gilbarg and Trudinger [18] and Kufner, John, and Fučik [20]. In (i) and (ii), we do not need the C^1 boundary requirement if $W^{1,p}(\Omega)$ is replaced by $W_0^{1,p}(\Omega)$.

Appendix B. The chain rule in $W^{1,p}(\Omega)$ and its subspaces

In this appendix we recall some important results from section 7.4 of Gilbarg and Trudinger [18] which still hold in $W^{1,p}(\Omega)$ and its subspaces, where Ω is a bounded domain in $\mathbb{R}^N$ and $p > 1$. In fact, we consider the chain rule in $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$, and $W_0^{1,p}(\Omega)$. The latter two spaces with the induced norm $\|\cdot\|_{W^{1,p}(\Omega)}$ are closed subspaces of $W^{1,p}(\Omega)$.

If $\partial\Omega$ is of class C^1 we can study the trace of a given u in $W^{1,p}(\Omega)$ which we denote by $u|_{\partial\Omega}$.

Lemma B.1. *Let f be in $C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$. Then:*

(i) *If $u \in W^{1,p}(\Omega)$, then $f \circ u \in W^{1,p}(\Omega)$.*
(ii) *If $u \in W_0^{1,p}(\Omega) \oplus \mathbb{R}$, then $f \circ u \in W_0^{1,p}(\Omega) \oplus \mathbb{R}$.*
(iii) *If $u \in W_0^{1,p}(\Omega)$ and $f(0) = 0$, then $f \circ u \in W_0^{1,p}(\Omega)$.*

In all cases we have $\nabla(f \circ u) = f'(u)\nabla u$. Moreover, the traces of u and $f(u)$ on $\partial\Omega$ satisfy

$$f(u|_{\partial\Omega}) = f(u)|_{\partial\Omega}.$$

The positive and negative parts of a function u are defined by

$$u^+(x) = \max\{u(x), 0\}, \ u^-(x) = \min\{u(x), 0\}.$$

Lemma B.2. *Let X be either one of the three spaces $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega) \oplus \mathbb{R}$, $W_0^{1,p}(\Omega)$. If $u \in X$, then $u^+, u^-, |u|$ are in X and*

$$\begin{aligned} \nabla u^+ &= \begin{cases} \nabla u, & \text{if } u > 0, \\ 0, & \text{if } u \le 0. \end{cases} \\ \nabla u^- &= \begin{cases} 0, & \text{if } u \ge 0, \\ \nabla u, & \text{if } u < 0. \end{cases} \\ \nabla |u| &= \begin{cases} \nabla u, & \text{if } u > 0, \\ 0, & \text{if } u = 0, \\ -\nabla u, & \text{if } u < 0. \end{cases} \end{aligned}$$

Furthermore, $(u|_{\partial\Omega})^+ = u^+|_{\partial\Omega}$ and $(u|_{\partial\Omega})^- = u^-|_{\partial\Omega}$.

We call a function piecewise smooth if it is continuous and has piecewise continuous first derivatives. The set of points at which f is not differentiable is called the set of corner points of f. The following chain rule generalizes the two previous lemmas.

Theorem B.3. *Let $f \in C(\mathbb{R})$ be a piecewise smooth function with $f' \in L^\infty(\mathbb{R})$. Then*

(i) *If $u \in W^{1,p}(\Omega)$, then $f \circ u \in W^{1,p}(\Omega)$.*
(ii) *If $u \in W_0^{1,p}(\Omega) \oplus \mathbb{R}$, then $f \circ u \in W_0^{1,p}(\Omega)$.*
(iii) *If $u \in W_0^{1,p}(\Omega)$ and $f(0) = 0$, then $f \circ u \in W_0^{1,p}(\Omega)$.*

In all cases, we have

$$\nabla(f\circ u)=\begin{cases} f'(u)\nabla u, & \text{if } u\notin L,\\ 0, & \text{if } u\in L,\end{cases}$$

where L denotes the set of corner points of f. Furthermore, $f(u|_{\partial\Omega}) = f(u)|_{\partial\Omega}$.

Appendix C. Restriction of functions to nodal domains

Let u be a continuous function on a given bounded domain Ω in $\mathbb{R}^N$. We define the zero set of u to be

$$Z(u)=\{x\in\Omega : u(x)=0\},$$

and call Ω_1 a **nodal domain** of u if Ω_1 is a component of $\Omega\setminus Z(u)$.

We recall here two important results from Brézis [6].

Lemma C.1. *(Theorem IX.17,* [6]*.)*

Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary. Given $u\in W^{1,p}(\Omega)\cap C(\bar{\Omega})$, $1\le p<\infty$, then the following are equivalent:

(i) *$u=0$ on $\partial\Omega$.*
(ii) *$u\in W_0^{1,p}(\Omega)$.*

In fact, (i)⇒(ii) does not require that $\partial\Omega$ be of class C^1 but the reverse implication does.

Lemma C.2. *(Proposition IX.18,* [6]*)*

Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary and let $u\in L^p(\Omega)$, $1<p<\infty$. The following are equivalent:

(i) *$u\in W_0^{1,p}(\Omega)$.*
(ii) *There exists a constant C such that*

$$\left|\int_\Omega u\frac{\partial\varphi}{\partial x_i}\right|\le C\|\varphi\|_{L^{p'}},\ \forall\varphi\in C_0^1(\mathbb{R}^N),\ i=1,\cdots,N,$$

where $1/p+1/p'=1$.

(iii) *The function*

$$\bar{u}=\begin{cases} u(x), & \text{if } x\in\Omega,\\ 0, & \text{if } x\in\mathbb{R}^N\setminus\Omega,\end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^N)$. In this case $\frac{\partial\bar{u}}{\partial x_i}=\overline{\frac{\partial u}{\partial x_i}}$.

Furthermore, the arguments that (i)⇒(ii)⇒(iii) do not require that $\partial\Omega$ be of class C^1 and still hold if $\mathbb{R}^N$ is replaced by an open set Ω' containing Ω.

We state the main result of this section.

Theorem C.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ and let X be either $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$. Suppose u is a function in $X\cap C(\Omega)$. Given Ω_1, a nodal domain of u, then the function u_1 defined by $u_1=u\chi_{\Omega_1}$, i.e.,*

$$u_1(x)=\begin{cases} u(x), & \text{for } x\in\Omega_1,\\ 0, & \text{for } \Omega\setminus\Omega_1,\end{cases}$$

is in X.

In the case $X = W_0^{1,p}(\Omega) \oplus \mathbb{R}$ we assume further that $\partial\Omega$ is connected and that u is in $C(\bar{\Omega}) \cap W_0^{1,p}(\Omega) \oplus \mathbb{R}$. Then for any nodal domain Ω_1 of u, the function u_1 defined by $u_1 = u\chi_{\overline{\Omega}_1}$, i.e.,

$$u_1(x) = \begin{cases} u(x), & \text{for } x \in \overline{\Omega}_1, \\ 0, & \text{for } \bar{\Omega} \setminus \overline{\Omega}_1, \end{cases}$$

is in $W_0^{1,p}(\Omega) \oplus \mathbb{R}$.

In all cases, we have $\nabla u_1(x) = \nabla u(x)\chi_{\Omega_1}(x)$ for a.e. $x \in \Omega$.

Proof. Since u is strictly positive or negative in Ω_1 we can assume $u > 0$ in Ω_1. Replacing u by u^+ we can also assume that $u \geq 0$ in Ω.

Case I: $X = W^{1,p}(\Omega)$.

The proof of this case is a modification of lemma 5.2 of De Figueiredo and Gossez [11].

We first show that u_1 belongs to $W^{1,p}(\Omega)$ in a neighborhood of every point x of Ω.

This is obvious if $x \in \Omega_1$ or $x \in \Omega \setminus \overline{\Omega}_1$. Let us consider the remaining possibility: $x \in \Omega \cap \partial\Omega_1$.

Taking an open ball B centered at x with $\bar{B} \subset \Omega$ and a function $\varphi \in C_0^\infty(B)$ with $\varphi = 1$ in a neighborhood of x, it suffices to show that $\varphi u_1 \in W_0^{1,p}(B)$. Call v the restriction of φu_1 to $B \cap \Omega_1$. Since $v \in C(\overline{B \cap \Omega_1}) \cap W^{1,p}(B \cap \Omega_1)$ vanishes on $\partial(B \cap \Omega_1)$, and by lemma C.1 v belongs to $W_0^{1,p}(B \cap \Omega_1)$. It follows from lemma C.2 that $\varphi u_1 \in W_0^{1,p}(B)$.

We have just shown that u_1 is in $W_{loc}^{1,p}(\Omega)$. However, by lemma C.2.(iii) and a direct computation one can easily see that $\nabla u_1(x) = \nabla u(x)\chi_{\Omega_1}(x)$ for a.e. $x \in \Omega$, which implies $|\nabla u_1(x)| \leq |\nabla u(x)|$, a.e. $x \in \Omega$. Therefore $u_1 \in W^{1,p}(\Omega)$.

Case II: $X = W_0^{1,p}(\Omega)$.

The following argument is contained in lemma 5.6, Cuesta, De Figueiredo, and Gossez [10].

Approximating u by a sequence of functions in $C_0^\infty(\Omega)$ and taking positive parts, we obtain a sequence $\{v_n\} \subset W_0^{1,p}(\Omega) \cap C(\Omega)$ with $v_n \geq 0$, supp v_n compact in Ω and $v_n \to u$ in $W_0^{1,p}(\Omega)$ (see lemma A.3). Let $w_n = \min\{u, v_n\}$, then the sequence $\{w_n\}$ has the same property as $\{v_n\}$. Since w_n has compact support, $w_n \in C(\bar{\Omega})$.

We claim that $w_n(x) = 0$ for all $x \in \partial\Omega_1$. Indeed, if $x \in \Omega \cap \partial\Omega_1$, then $u(x) = 0$ (since Ω_1 is a nodal domain) and thus $w_n(x) = \min\{u(x), v_n(x)\} = 0$. If $x \in \partial\Omega \cap \partial\Omega_1$, then $w_n(x) = 0$ since w_n has compact support.

For each n, we define $\overline{w_n}(x) = w_n(x)\chi_{\Omega_1}(x)$, $x \in \bar{\Omega}$. Then case I implies that $\overline{w_n} \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. Since $w_n = 0$ on $\partial\Omega_1$, $\overline{w_n} = 0$ on $\partial\Omega$. Thus, by lemma C.1 we have $\overline{w_n} \in W_0^{1,p}(\Omega)$. Moreover, since $\nabla u_1 = \nabla u \chi_{\Omega_1}$, $\nabla\overline{w_n} = \nabla w_n \chi_{\Omega_1}$ and $w_n \to u$ in $W^{1,p}(\Omega)$, we conclude that $\overline{w_n} \to u_1$ in $W^{1,p}(\Omega)$. Therefore, u_1 is in $W_0^{1,p}(\Omega)$.

Case III: $X = W_0^{1,p}(\Omega) \oplus \mathbb{R}$.

Case I implies that u_1 is in $W^{1,p}(\Omega)$. Let $u|_{\partial\Omega} \equiv c$ be the constant value of u on $\partial\Omega$. By the definition of u_1, either $u_1(x) = 0$ or $u_1(x) = c$ for any $x \in \partial\Omega$. However, since $u_1 \in C(\bar{\Omega})$ and $\partial\Omega$ is connected, either $u_1|_{\partial\Omega} \equiv 0$ or $u_1|_{\partial\Omega} \equiv c$. Therefore, u_1 belongs to in $W_0^{1,p}(\Omega) \oplus \mathbb{R}$. □

We have similar results for the trace of restriction functions.

Lemma C.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary. Let u in $C^1(\bar{\Omega}) \cap W^{1,p}(\Omega)$ be such that it has only finitely many nodal domains. Then for any nodal domain Ω_0 of u, the function u_0 defined by $u_0 = u\chi_{\overline{\Omega_0}}$ is in $W^{1,p}(\Omega)$ and the trace $u_0|_{\partial\Omega}$ of u_0 satisfies*

$$u_0|_{\partial\Omega}(x) = \begin{cases} u(x), & x \in \partial\Omega \cap \partial\Omega_0, \\ 0, & x \in \partial\Omega \setminus \partial\Omega_0. \end{cases} \tag{C.1}$$

Proof. The fact that u_0 is in $W^{1,p}(\Omega)$ follows from the previous theorem. To verify (C.1) we may assume without loss of generality that $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ consists of two components Ω_1 and Ω_2.

Let $u_1 = u\chi_{\overline{\Omega_1}}$ and $u_2 = u\chi_{\overline{\Omega_2}}$. We will show that the traces of u_1 and u_2 satisfy formula (C.1) (with respect to the corresponding nodal domain). By extending u to a function in $C_0^1(B)$, where B is a ball containing $\bar{\Omega}$, we can find two sequences $\{v_n\}$ and $\{w_n\}$ in $C^1(\bar{\Omega})$ such that supp $v_n \subset \overline{\Omega}_1$, supp $w_n \subset \overline{\Omega}_2$ for any n and $v_n \to u_1$, $w_n \to u_2$ in $W^{1,p}(\Omega)$. Consequently, the traces $v_n|_{\partial\Omega}$, $w_n|_{\partial\Omega}$ converge to $u_1|_{\partial\Omega}$ and $u_2|_{\partial\Omega}$ in $L^p(\partial\Omega)$, respectively. On the other hand, we notice that $v_n + w_n$ converges to u^+ in $W^{1,p}(\Omega)$, and thus $(v_n|_{\partial\Omega} + w_n|_{\partial\Omega})$ converges to $u^+|_{\partial\Omega}$ in $L^p(\partial\Omega)$.

By lemma B.2, $u^+|_{\partial\Omega}$ is the restriction of u^+ to $\partial\Omega$. i.e.,

$$u^+|_{\partial\Omega}(x) = \begin{cases} u(x), & x \in \partial\Omega \cap \partial\Omega_1 \text{ or } x \in \partial\Omega \cap \partial\Omega_2, \\ 0, & \text{otherwise }. \end{cases}$$

We recall that the traces $v_n|_{\partial\Omega}$, $w_n|_{\partial\Omega}$ are just the restrictions of v_n and w_n to $\partial\Omega$, respectively. Since supp$v_n \subset \overline{\Omega}_1$, supp$w_n \subset \overline{\Omega}_2$ and Ω_1, Ω_2 are disjoint, we conclude that

$$u_i|_{\partial\Omega}(x) = \begin{cases} u(x), & x \in \partial\Omega \cap \partial\Omega_i, \\ 0, & \text{otherwise }, \end{cases}$$

for $i = 1, 2$.

□

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References

[1] R. Adams, *Sobolev Spaces*, Academic Press, NewYork, 1975.

[2] A. Anane, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, C. R. Acad. Sci. Paris Sér. I Math., 305 (1987), pp. 725–728.

[3] A. Anane and N. Tsouli, *On the second eigenvalue of the p-Laplacian*, in Nonlinear Partial Differential Equations (Fès, 1994), vol. 343 of Pitman Res. Notes Math. Ser., Longman, Harlow, 1996, pp. 1–9.

[4] G. Astarita and G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics*, McGraw-Hill, 1974.

[5] J. F. Bonder and J. D. Rossi, *Existence results for the p-Laplacian with nonlinear boundary conditions*, J. Math. Anal. Appl., 263 (2001), pp. 195–223.

[6] H. Brézis, *Analyse Fonctionnelle: Théorie et Applications*, Masson, Paris, 1983.

[7] F. Browder, *Existence theorems for nonlinear partial differential equations*, in Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 1–60.

[8] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I*, Interscience Publishers, Inc., New York, N.Y., 1953.

[9] ———, *Methods of Mathematical Physics. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989. Partial Differential Equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

[10] M. Cuesta, D. de Figueiredo, and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations, 159 (1999), pp. 212–238.

[11] D. G. de Figueiredo and J.-P. Gossez, *On the first curve of the Fučik spectrum of an elliptic operator*, Differential Integral Equations, 7 (1994), pp. 1285–1302.

[12] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., 7 (1983), pp. 827–850.

[13] P. Drábek, A. Kufner, and F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, vol. 5 of de Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter & Co., Berlin, 1997.

[14] P. Drábek and R. Manásevich, *On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian*, Differential Integral Equations, 12 (1999), 773–788.

[15] L. Friedlander, *Asymptotic behavior of the eigenvalues of the p-Laplacian*, Comm. Partial Differential Equations, 14 (1989), pp. 1059–1069.

[16] S. Fučík, J. Nečas, J. Souček, and V. Souček, *Spectral Analysis of Nonlinear Operators*, Springer-Verlag, Berlin, 1973. Lecture Notes in Mathematics, Vol. 346.

[17] J. P. García Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues*, Comm. Partial Differential Equations, 12 (1987), pp. 1389–1430.

[18] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[19] T. Iwaniec and J. J. Manfredi, *Regularity of p-harmonic functions on the plane*, Rev. Mat. Iberoamericana, 5 (1989), pp. 1–19.

[20] A. Kufner, O. John, and S. Fučik, *Function Spaces*, Noordhoff, Leyden, 1977.

[21] V. K. Le and K. Schmitt, *Global Bifurcation in Variational Inequalities: Applications to Obstacle and Unilateral Problems*, Springer-Verlag, New York, 1997.

[22] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., 12 (1988), pp. 1203–1219.

[23] P. Lindqvist, *Addendum: "On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$*" [Proc. Amer. Math. Soc.* **109** *(1990), 157–164]*, Proc. Amer. Math. Soc., 116 (1992), pp. 583–584.

[24] S. Martínez and J. D. Rossi, *Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition*, Abstr. Appl. Anal., 7 (2002), pp. 287–293.

[25] I. Nečas, *The discreteness of the spectrum of a nonlinear Sturm-Liouville equation*, Soviet Math. Dokl., 12 (1971), pp. 1779–1783.

[26] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, vol. 146 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, 1991.
[27] K. Schmitt and I. Sim, *Bifurcation problems associated with generalized Laplacians*, Adv. Differential Equations, 9 (2004), pp. 797–828.
[28] R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Electronic Monographs in Differential Equations, San Marcos, TX, 1994. Electronic reprint of the 1977 original.
[29] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, 51 (1984), pp. 126–150.
[30] S. L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math., 37 (1970/71), pp. 173–180.
[31] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math., 20 (1967), pp. 721–747.
[32] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math., 138 (1977), pp. 219–240.
[33] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., 12 (1984), pp. 191–202.
[34] E. Zeidler, *The Ljusternik-Schnirelman theory for indefinite and not necessarily odd nonlinear operators and its applications*, Nonlinear Anal., 4 (1980), pp. 451–489.
[35] ———, *Nonlinear Functional Analysis and its Applications, Vol.3: Variational Methods and Optimization*, Springer, Berlin, 1985.
[36] ———, *Nonlinear Functional Analysis and its Applications, Vol.2A: Linear Monotone Operators*, Springer, Berlin, 1990.

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