

# Variational eigenvalues of degenerate eigenvalue problems for the weighted $p$ -Laplacian

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## Abstract

We prove the existence of nondecreasing sequences of positive eigenvalues of the homogeneous degenerate quasilinear eigenvalue problem  $-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda b(x)|u|^{p-2}u$ ,  $\lambda > 0$  with Dirichlet boundary condition on a bounded domain  $\Omega$ . The diffusion coefficient  $a(x)$  is a function in  $L^1_{loc}(\Omega)$  and  $b(x)$  is a nontrivial function in  $L^r(\Omega)$  ( $r$  depending on  $a$ ,  $p$  and  $N$ ) and may change sign. We use Ljusternik-Schnirelman theory, minimax theory and the theory of weighted Sobolev spaces to establish our results.

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## 1 Introduction

In this work we study the homogeneous degenerate eigenvalue problem

$$\begin{cases} -\operatorname{div}((a(x)|\nabla u|^{p-2}\nabla u) & = \lambda b(x)|u|^{p-2}u, \text{ in } \Omega, \\ u & = 0, \text{ on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  and  $1 < p \leq N$ . We assume that

$$\begin{aligned} a(x) &\text{ is positive a.e. in } \Omega, \\ a &\in L^1_{loc}(\Omega), \quad a^{-s} \in L^1(\Omega), \quad s \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right). \end{aligned} \quad (1.2)$$

For example, if  $a(x) = 1/\text{dist}(x, \partial\Omega)$ , then  $a$  satisfies (1.2).

With the number  $s$  given in (1.2) we define

$$p_s = \frac{ps}{s+1}, \quad p_s^* = \frac{Np_s}{N-p_s} = \frac{Nps}{N(s+1)-ps} > p. \quad (1.3)$$

We assume further that

$$\begin{aligned} \text{meas}\{x \in \Omega : b(x) > 0\} &> 0, \\ b &\in L^{\frac{q}{q-p}}(\Omega), \quad \text{for some } p \leq q < p_s^*. \end{aligned} \quad (1.4)$$

If  $q = p$  in (1.4) then  $b$  is assumed to be in  $L^\infty(\Omega)$ .

The problem (1.1) was introduced by Drábek, Kufner and Nicolosi in [7] (Chapter 3) and it was also considered (as a special case) in Le and Schmitt [13]. (1.1) is called *degenerate* (and/or *singular*) to distinguish it from the homogeneous *non-degenerate* eigenvalue problem for the  $p$ -Laplacian

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) &= \lambda|u|^{p-2}u, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega, \end{cases} \quad (1.5)$$

which has been studied extensively during the past two decades. Many interesting results have been obtained for (1.5), see, e.g., García Azorero and Peral Alonso [10], Anane [1], Lindqvist [15], Anane and Toulí [2], Drábek and Robinson [8, 9], etc... In a forthcoming paper, Lê [12] gives a detailed summary of the results covering the Dirichlet problem (1.5) and studies other eigenvalue problems subject to different kinds of boundary conditions.

This work is motivated by results in the book of Drábek, Kufner and Nicolosi [7] in Le and Schmitt [13] and in Cuesta [5]. In Chapter 3 of [7], the authors use a variational method in weighted Sobolev spaces to show the existence of a principal eigenvalue (the least positive eigenvalue) of (1.1) and corresponding nonnegative eigenvalues. In Section 7 of [13], a similar method (in different weighted Sobolev spaces) was used to establish the existence of the principal eigenvalue. Using information about this eigenvalue, the authors in [13] deduced existence results about nontrivial solutions to fully nonlinear problems. In Cuesta [5] the author studied positive eigenvalues of the problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) &= \lambda b(x)|u|^{p-2}u, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega, \end{cases} \quad (1.6)$$

where  $b \in L^r(\Omega)$ ,  $r > \frac{N}{p}$  and  $b$  changes sign. She established among other results the existence of sequences of positive eigenvalues and showed that the principal

eigenvalue  $\lambda_1$  is simple and isolated. In both problems (1.1) and (1.6) the principal eigenvalue is characterized as

$$\lambda_1 := \inf \left\{ \int_{\Omega} a |\nabla u|^p dx : u \in W_0^{1,p}(a, \Omega) \text{ and } \int_{\Omega} b |u|^p dx = 1 \right\},$$

where  $W_0^{1,p}(a, \Omega)$  is a weight Sobolev space ( $W_0^{1,p}(a, \Omega) \equiv W_0^{1,p}(\Omega)$  when  $a$  is a positive constant function) defined in Section 2.

The purpose of this paper is to study positive eigenvalues of problem (1.1). As mentioned in [5] and in [8], there are two methods in establishing sequences of eigenvalues. One is Ljusternik-Schnirelman theory and the other is minimax theory. Even though there are many versions and many forms of Ljusternik-Schnirelman principle and of minimax theory, applications of these theories usually take place in regular Sobolev spaces  $W^{1,p}(\Omega)$  and their subspaces (cf. [5, 8, 10, 12, 16]). In this paper, we will see that such theories can be applied in weighted Sobolev spaces.

We shall use both methods (Ljusternik-Schnirelman principle and minimax theory) to study (1.1). For the Ljusternik-Schnirelman principle approach, we follow the arguments developed in Section 2 of [12]. To do so we need to verify the hypotheses given by the theory in the underlying weighted Sobolev space  $W_0^{1,p}(a, \Omega)$ . These verifications require some technical calculations. For the minimax theory approach, we use a minimax theorem on  $C^1$  manifolds proved by Cuesta (Proposition 2.7, [6]). The main difficulty is to show  $G(u) = \int_{\Omega} a |\nabla u|^p dx$  satisfies the Palais-Smale condition on a given  $C^1$  manifold. Our main results are given in Theorem 3.2 and Theorem 4.1.

We notice that if  $a \equiv 1$  then  $s = \infty$ ; thus  $p_s = p$  and  $\frac{q}{q-p} = r > \frac{N}{p}$ . It follows that (1.6) is a special case of (1.1). Consequently, our results extend results in [5] in weighted Sobolev spaces.

The rest of the paper is organized as follows. In Section 2 we recall the definition of weighted Sobolev spaces and state basic properties of such spaces. In Section 3 we state a Ljusternik-Schnirelman principle and use it to establish the existence of a nondecreasing sequence of positive eigenvalues of problem (1.1). In Section 4 we apply minimax theory to establish the existence of another nondecreasing sequence of positive eigenvalues of problem (1.1). Finally in Section 5 we give a comparison between the two sequences of eigenvalues established in Section 3 and Section 4.

We remark that the assumption  $1 < p \leq N$  is made only for the purpose of simplicity. When  $p > N$ , we obtain the same results if we modify conditions on parameters appropriately.

## 2 Preliminaries

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  and  $1 < p < N$  be a real number. In what follows we will briefly recall some definitions, concepts and basic results for weighted Sobolev spaces. For details and proofs, we refer to Drábek et al [7] (Section 1.5) and other references therein.

Given  $a(x)$  satisfying (1.2), the weighted Sobolev  $W^{1,p}(a, \Omega)$  is defined to be the set of all real valued measurable functions  $u$  for which

$$\|u\|_{1,p,a} = \left( \int_{\Omega} |u|^p dx + \int_{\Omega} a |\nabla u|^p dx \right)^{1/p} < \infty. \quad (2.1)$$

Then  $W^{1,p}(a, \Omega)$  equipped with the norm  $\|\cdot\|_{1,p,a}$  is a uniformly convex Banach space; thus, by Milman's Theorem (see [17]) it is a reflexive Banach space. Moreover, the continuous embedding

$$W^{1,p}(a, \Omega) \hookrightarrow W^{1,p_s}(\Omega) \quad (2.2)$$

holds with  $p_s = \frac{ps}{s+1}$  (cf. Example 1.3, [7]).

Let  $X := W_0^{1,p}(a, \Omega)$  be the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p}(a, \Omega)$  with respect to the norm  $\|\cdot\|_{1,p,a}$ . Then  $X$  is a uniformly convex Banach space, as well. We conclude from the embedding (2.2) and Sobolev's embedding theorem that

$$X \hookrightarrow W_0^{1,p_s}(\Omega) \hookrightarrow L^{p_s^*}(\Omega), \quad (2.3)$$

where

$$p_s^* := \frac{Np_s}{N-p_s} = \frac{Nps}{N(s+1)-ps} > p.$$

We notice that the compact embedding

$$X \hookrightarrow L^r(\Omega) \quad (2.4)$$

holds provided that  $1 \leq r < p_s^*$ .

It follows from the weighted Friedrichs inequality (see [7] (formula (1.28), page 27)) that the norm

$$\|u\| = \left( \int_{\Omega} a |\nabla u|^p dx \right)^{1/p} \quad (2.5)$$

on the space  $X$  is equivalent to the norm  $\|\cdot\|_{1,p,a}$  defined in (2.1).

**Definition 2.1** We say  $\lambda \in \mathbb{R}^+$  is a positive eigenvalue of (1.1) if there exists a nontrivial function  $u$  in  $X$  such that

$$\int_{\Omega} a(x) |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_{\Omega} b(x) |u|^{p-2} u v dx \quad (2.6)$$

holds for any  $v \in X$ . Such a function  $u$  is called an eigenfunction corresponding to the eigenvalue  $\lambda$ . The pair  $(u, \lambda)$  is called an eigenpair.

The following theorem has been proved in Chapter 3 of [7].

**Theorem 2.1** *Problem (1.1) has a least positive (i.e., a first or principal) eigenvalue  $\lambda_1 > 0$ . It is a simple eigenvalue (i.e., any two eigenfunctions associated with  $\lambda_1$  differ by a constant multiple) and all eigenfunctions corresponding to  $\lambda_1$  do not change sign in  $\Omega$ .*

In fact,  $\lambda_1$  is characterized as

$$\lambda_1 = \inf_{u \in K} \int_{\Omega} a(x) |\nabla u|^p dx, \quad (2.7)$$

where  $K = \{u \in X : \int_{\Omega} b(x) |u|^p dx = 1\}$  and  $X = W_0^{1,p}(a, \Omega)$ .

**Remark 2.1** As mentioned in Section 1, any function  $a(x) = \frac{1}{\text{dist}(x, \partial\Omega)^\alpha}$ ,  $\alpha \geq 0$ , satisfies (1.2); it means, in this case, that the diffusion coefficient can blow up near the boundary.

**Remark 2.2** Problem (1.1) is a generalization of the regular Dirichlet problem (1.5) and problem (1.6). When  $a \equiv 1$ , we obtain the spaces  $W^{1,p}(\Omega)$  and  $W_0^{1,p}(\Omega)$  with  $s = \infty$  and  $p_s^* = p^* = Np/(n-p)$ .

**Remark 2.3** Weighted Sobolev spaces defined in [13] and defined by (2.1) are generally different. However they coincide if the weight  $a$  satisfies some properties, see e.g. [13] and [11].

### 3 Ljusternik-Schnirelman theory

In this section, we recall a version of the Ljusternik-Schnirelman principle which was discussed by F. Browder [4] and E. Zeidler [18], [19] (Section 44.5, remark 44.23). We then shall apply the principle (Theorem 3.1) to establish the existence of a sequence of positive eigenvalues of problem (1.1).

Let  $X$  be a real reflexive Banach space and  $F, G$  be two functionals on  $X$ . Consider the following eigenvalue problem

$$F'(u) = \mu G'(u), \quad u \in S, \quad \mu \in \mathbb{R}, \quad (3.1)$$

where  $S$  is the level  $S := \{u \in X : G(u) = 1\}$ . We assume that:

**(H1)**  $F, G : X \rightarrow \mathbb{R}$  are even functionals and that  $F, G \in C^1(X, \mathbb{R})$  with  $F(0) = G(0) = 0$ .

**(H2)**  $F'$  is strongly continuous (i.e.  $u_n \rightharpoonup u$  in  $X$  implies  $F'(u_n) \rightarrow F'(u)$ ) and  $\langle F'(u), u \rangle = 0$ ,  $u \in \overline{coS}$  implies  $F(u) = 0$ , where  $\overline{coS}$  is the closed convex hull of  $S$ .

**(H3)**  $G'$  is continuous, bounded and satisfies condition  $(S_0)$ , i.e. as  $n \rightarrow \infty$ ,

$$u_n \rightharpoonup u, \quad G'(u_n) \rightharpoonup v, \quad \langle G'(u_n), u_n \rangle \rightarrow \langle v, u \rangle \text{ implies } u_n \rightarrow u.$$

**(H4)** The level set  $S$  is bounded and  $u \neq 0$  implies

$$\langle G'(u), u \rangle > 0, \quad \lim_{t \rightarrow +\infty} G(tu) = +\infty \text{ and } \inf_{u \in S} \langle G'(u), u \rangle > 0.$$

It is known that  $(u, \mu)$  solves (3.1) if and only if  $u$  is a critical point of  $F$  with respect to  $S$  (see Zeidler [19], Proposition 43.21).

Let

$\Sigma := \{K : K \text{ is a compact, symmetric subset of } S \text{ such that } F(u) > 0, u \in K\}$ .

For any positive integer  $n$ , define

$$\Sigma_n := \{K \subset \Sigma : g(K) \geq n\}, \quad (3.2)$$

where  $g(K)$  denotes the genus of  $K$ , i.e.,  $g(K) := \inf\{k \in \mathbb{N} : \exists h : K \rightarrow \mathbb{R}^k \setminus \{0\} \text{ such that } h \text{ is continuous and odd}\}$ .

$$\beta_n := \begin{cases} \sup_{K \in \Sigma_n} \inf_{u \in K} F(u), & \Sigma_n \neq \emptyset. \\ 0, & \Sigma_n = \emptyset. \end{cases} \quad (3.3)$$

Also let

$$\chi := \begin{cases} \sup\{n \in \mathbb{N} : \beta_n > 0\}, & \text{if } \beta_1 > 0, \\ 0, & \text{if } \beta_1 = 0. \end{cases} \quad (3.4)$$

We now state the Ljusternik-Schnirelman principle.

**Theorem 3.1** *Under assumptions (H1)-(H4), the following assertions hold:*

- (1) *If  $\beta_n > 0$ , then (3.1) possesses a pair  $\pm u_n$  of eigenvectors and an eigenvalue  $\mu_n \neq 0$ ; furthermore  $F(u_n) = \beta_n$ .*
- (2) *If  $\chi = \infty$ , (3.1) has infinitely many pairs  $\pm u$  of eigenvectors corresponding to nonzero eigenvalues.*
- (3)  *$\infty > \beta_1 \geq \beta_2 \geq \dots \geq 0$  and  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$ .*
- (4) *If  $\chi = \infty$  and  $F(u) = 0, u \in \overline{coS}$  implies  $\langle F'(u), u \rangle = 0$ , then there exists an infinite sequence  $\{\mu_n\}$  of distinct eigenvalues of (3.1) such that  $\mu_n \rightarrow 0$  as  $n \rightarrow \infty$ .*
- (5) *Assume that  $F(u) = 0, u \in \overline{coS}$  implies  $u = 0$ . Then  $\chi = \infty$  and there exists a sequence of eigenpairs  $\{(u_n, \mu_n)\}$  of (3.1) such that  $u_n \rightarrow 0, \mu_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\mu_n \neq 0$  for all  $n$ .*

*Proof.* We refer to Browder [4] or Zeidler [18] for the proof.

**Remark 3.1** To obtain a sequence of negative eigenvalues we replace  $F$  with  $-F$ .

We are going to use Theorem 3.1 to study (1.1). Let  $a$  and  $b$  be functions satisfying (1.2) and (1.4), respectively. As mentioned in Section 2 the space  $X = W_0^{1,p}(a, \Omega)$  is a reflexive Banach space. We define on  $X$  two functionals

$$F(u) = \int_{\Omega} b|u|^p dx, \quad (3.5)$$

$$G(u) = \int_{\Omega} a|\nabla u|^p dx = \|u\|^p, \text{ (see (2.5)).} \quad (3.6)$$

Let  $S = \{u \in X : G(u) = 1\}$ . It is easy to see that  $F$  and  $G$  are differentiable with  $A = \frac{1}{p}F'$  and  $B = \frac{1}{p}G'$  given by

$$\langle Au, v \rangle = \int_{\Omega} b|u|^{p-2}uv dx, \quad (3.7)$$

$$\langle Bu, v \rangle = \int_{\Omega} a|\nabla u|^{p-2}\nabla u \cdot \nabla v dx. \quad (3.8)$$

Then (3.1) becomes  $Au = \mu Bu$ , i.e., for any  $v \in X$ ,

$$\int_{\Omega} b|u|^{p-2}uv dx = \mu \int_{\Omega} a|\nabla u|^{p-2}\nabla u \cdot \nabla v dx. \quad (3.9)$$

We claim that  $F$  and  $G$  satisfy hypotheses (H1), (H2), (H3), and (H4). Since  $G(u) = \|u\|^{1/p}$ , (H4) holds. It is clear that  $F$  and  $G$  are even and thus it remains to verify (H2) and (H3). In Proposition 3.1 below we show that  $F'$  satisfies (H2) by using Sobolev's embedding (2.3) and Hölder's inequality. The proof is a modification of the argument made in Proposition 2.2, Lê [12].

**Proposition 3.1** *Let  $F$  be defined in (3.5), then  $F'$  satisfies (H2).*

*Proof.* It suffices to show that  $A$  is strongly continuous. Let  $u_n \rightharpoonup u$  in  $X$ , we need to show that  $Au_n \rightarrow Au$  in  $X^*$ .

For any  $v \in X$ , by Hölder's inequality and Sobolev's embedding (2.3) it follows that

$$\begin{aligned} |\langle Au_n - Au, v \rangle| &= \left| \int_{\Omega} b(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx \right| \\ &\leq \|b\|_{\alpha} \| |u_n|^{p-2}u_n - |u|^{p-2}u \|_{\frac{\beta}{p-1}} \|v\|_{p_s^*} \\ &\leq C \|b\|_{\alpha} \| |u_n|^{p-2}u_n - |u|^{p-2}u \|_{\frac{\beta}{p-1}} \|v\|. \end{aligned}$$

where  $\alpha, \beta$  are such that  $\frac{1}{\alpha} + \frac{p-1}{\beta} + \frac{1}{p_s^*} = 1$ , and  $\|\cdot\|, \|\cdot\|_p$  are the norms in  $X$  and in  $L^p(\Omega)$ , respectively. We observe that

$$\frac{p_s^* - p}{p_s^*} + \frac{p-1}{p_s^*} + \frac{1}{p_s^*} = 1. \quad (3.10)$$

Since  $b$  is in  $L^{\frac{q}{q-p}}(\Omega)$  (condition (1.4)) and  $\frac{p_s^*}{p_s^* - p} < \frac{q}{q-p}$ , whenever  $p < q < p_s^*$ , we can choose  $\alpha$  such that  $\frac{p_s^*}{p_s^* - p} < \alpha < \frac{q}{q-p}$ . With this choice of  $\alpha$ , it follows from (3.10) that  $1 < \beta < p_s^*$ . We next show that  $|u_n|^{p-2}u_n \rightarrow |u|^{p-2}u$  in  $L^{\frac{\beta}{p-1}}(\Omega)$ . To see this, let  $w_n = |u_n|^{p-2}u_n$  and  $w = |u|^{p-2}u$ . Since  $u_n \rightharpoonup u$  in  $X$ ,  $u_n \rightarrow u$  in  $L^{\beta}(\Omega)$  by (2.4). It follows that

$$w_n(x) \rightarrow w(x), \text{ a.e. in } \Omega \quad \text{and} \quad \int_{\Omega} |w_n|^{\frac{\beta}{p-1}} dx \rightarrow \int_{\Omega} |w|^{\frac{\beta}{p-1}} dx.$$

We conclude that  $w_n \rightarrow w$  in  $L^{\frac{\beta}{p-1}}(\Omega)$ . Therefore  $Au_n \rightarrow Au$  in  $X^*$ . In order to verify (H3) we need the following lemma which uses a calculation from Chapter 6 of Le and Schmitt [14].

**Lemma 3.1** *Let  $B$  be defined in (3.8), then for any  $u$  and  $v$  in  $X$  one has*

$$\langle Bu - Bv, u - v \rangle \geq (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).$$

Furthermore,  $\langle Bu - Bv, u - v \rangle = 0$  if and only if  $u = v$  a.e. in  $\Omega$ .

*Proof.* Straightforward computations give us

$$\begin{aligned} \langle Bu - Bv, u - v \rangle &= \int_{\Omega} [a|\nabla u|^p + a|\nabla v|^p] dx \\ &\quad - \int_{\Omega} a|\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \int_{\Omega} a|\nabla v|^{p-2} \nabla v \cdot \nabla u dx \\ &= \|u\|^p + \|v\|^p \\ &\quad - \int_{\Omega} a|\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \int_{\Omega} a|\nabla v|^{p-2} \nabla v \cdot \nabla u dx. \end{aligned}$$

As the function  $a$  is positive, it follows from Hölder's inequality that

$$\begin{aligned} \int_{\Omega} a|\nabla u|^{p-2} \nabla u \cdot \nabla v dx &\leq \left( \int_{\Omega} a|\nabla u|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} a|\nabla v|^p dx \right)^{\frac{1}{p}} \quad (3.11) \\ &= \|u\|^{p-1} \|v\|, \end{aligned}$$

for any  $u, v$  in  $X$ . Thus,

$$\begin{aligned} \langle Bu - Bv, u - v \rangle &\geq \|u\|^p + \|v\|^p - \|u\|^{p-1} \|v\| - \|v\|^{p-1} \|u\| \\ &= (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|). \end{aligned}$$

Now let  $u$  and  $v$  be such that  $\langle Bu - Bv, u - v \rangle = 0$ . Then we have

$$\langle Bu - Bv, u - v \rangle = (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) = 0.$$

It follows that  $\|u\| = \|v\|$  and that the equality holds in (3.11). As equality in Hölder's inequality is characterized, we obtain that  $\nabla u = k\nabla v$  a.e. in  $\Omega$ , for some constant  $k \geq 0$ , which implies  $\|u\| = k\|v\|$ . Therefore,  $k = 1$  and  $u = v$  a.e. in  $\Omega$ . In what follows we use ideas similar to those developed in Lê [12].

**Proposition 3.2** *Let  $G$  be defined in (3.6) then  $G'$  satisfies (H3).*

*Proof.* As  $B = G'/p$ , it suffices to work with  $B$ . It is easy to see that  $B$  is bounded. To show the continuity of  $B$  we notice from Hölder inequality that

$$\begin{aligned} |\langle Bu_n - Bu, v \rangle| &= \left| \int_{\Omega} a(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot \nabla v dx \right| \\ &\leq \left( \int_{\Omega} \left| a^{\frac{p-1}{p}} |\nabla u_n|^{p-2} \nabla u_n - a^{\frac{p-1}{p}} |\nabla u|^{p-2} \nabla u \right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \|v\|, \end{aligned}$$

where  $\{u_n\}$  is a sequence converging to  $u$  in  $X$ . Let  $w_n = a^{\frac{p-1}{p}} |\nabla u_n|^{p-2} \nabla u_n$  and  $w = a^{\frac{p-1}{p}} |\nabla u|^{p-2} \nabla u$  then

$$w_n(x) \rightarrow w(x), \text{ a.e. in } \Omega \text{ and } \int_{\Omega} |w_n|^{\frac{p}{p-1}} dx \rightarrow \int_{\Omega} |w|^{\frac{p}{p-1}} dx.$$

Thus  $w_n \rightarrow w$  in  $L^{\frac{p}{p-1}}(\Omega)$  which proves the continuity of  $B$ .

It remains to show that  $B$  satisfies condition  $(S_0)$ . That means if  $\{u_n\}$  is a sequence in  $X$  such that

$$u_n \rightharpoonup u, \quad Bu_n \rightharpoonup v, \quad \text{and } \langle Bu_n, u_n \rangle \rightarrow \langle v, u \rangle$$

for some  $v \in X^*$  and  $u \in X$ , then  $u_n \rightarrow u$  in  $X$ .

Since  $X$  is a uniformly convex Banach space, weak convergence and norm convergence imply (strong) convergence. Thus to show  $u_n \rightarrow u$  in  $X$ , we only need to show  $\|u_n\| \rightarrow \|u\|$ . To this end, we first observe that

$$\lim_{n \rightarrow \infty} \langle Bu_n - Bu, u_n - u \rangle = \lim_{n \rightarrow \infty} (\langle Bu_n, u_n \rangle - \langle Bu_n, u \rangle - \langle Bu, u_n - u \rangle) = 0.$$

On the other hand, it follows from Lemma 3.1 that

$$\langle Bu_n - Bu, u_n - u \rangle \geq (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|).$$

Hence  $\|u_n\| \rightarrow \|u\|$  as  $n \rightarrow \infty$ . Therefore  $B$  satisfies condition  $(S_0)$ .

We now can apply Theorem 3.1 to conclude the following.

**Theorem 3.2 (Existence of L-S sequence)** *Let  $F$  and  $G$  be the two functionals defined in (3.5), (3.6). Then there exists a nonincreasing sequence of positive eigenvalues  $\{\mu_n\}$  obtained from the Ljusternik-Schnirelman principle such that  $\mu_n \rightarrow 0$  as  $n \rightarrow \infty$ , where*

$$\mu_n = \sup_{H \in \mathbb{A}_n} \inf_{u \in H} F(u), \quad (3.12)$$

and each  $\mu_n$  is an eigenvalue of  $F'(u) = \mu G'(u)$ .

*Proof.* Since  $S$  is the unit sphere in  $X$ , it contains compact subsets of arbitrarily large genus. Thus  $\Sigma_n$  is nonempty for any  $n$  (recall (3.2)). For instance,  $X_n \cap S$  is in  $\Sigma_n$  whenever  $X_n$  is a  $n$ -dimensional subspace of  $X$ . Given a set  $K$  in  $\Sigma_n$ , since  $K$  is compact and  $F$  is positive on  $K$ ,  $\inf_{u \in K} F(u) > 0$ . It follows that  $\beta_n > 0$ , where  $\beta_n$  is the critical value defined by (3.3). We conclude from (3.4) that  $\chi = \infty$ . The existence of such a sequence  $\{\mu_n\}$  follows from Theorem 3.1-(1), (2) and (3). To verify (3.12) we observe, using (3.5), (3.6), (3.7) and (3.8), that

$$\mu_n = \mu_n G(u_n) = \mu_n \langle Bu_n, u_n \rangle = \langle Au_n, u_n \rangle = F(u_n) = \beta_n.$$

Combining this with (3.3) we obtain (3.12).

The following corollary follows immediately from (2.6), (3.9) and Theorem 3.2.

**Corollary 3.1** *The sequence  $\{\lambda_n := \frac{1}{\mu_n}\}$  is a nondecreasing sequence of positive real numbers such that  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Moreover each  $\lambda_n$  is a positive eigenvalue of (1.1).*

**Remark 3.2** Assertions (4) and (5) in Theorem 3.1 do not hold. However, the fact that  $\mu_n \rightarrow 0$  is due to the homogeneity of  $F'$  and  $G'$ .

## 4 Minimax theory

In this section we adapt all definitions and notations from the previous sections. We recall that

$$X = W_0^{1,p}(a, \Omega), \quad F(u) = \int_{\Omega} b|u|^p dx, \quad G(u) = \int_{\Omega} a|\nabla u|^p dx.$$

Since  $F$  is a  $C^1$  functional and  $b^+ \neq 0$  (see (1.4)), the set  $M := \{u \in X : F(u) = 1\}$  is a nonempty  $C^1$  manifold. We denote by  $TG$  the restriction of  $G'$  to  $M$ , i.e. for each  $u \in M$  we have  $TG(u)(v) = \langle G'(u), v \rangle$  for all  $v \in T_u M$ , where  $T_u M := \{v \in X : \langle F'(u), v \rangle = 0\}$  is the tangent space of  $M$  at  $u$ . Thus  $\|TG(u)\| = \|TG(u)\|_{(T_u M)^*}$ .

**Definition 4.1**  $G$  is said to satisfy  $(PS)_{c,M}$ , the Palais-Smale condition on  $M$  at level  $c$ , if any sequence  $u_n \in M$  such that  $G(u_n) \rightarrow c$  and  $\|TG(u_n)\| \rightarrow 0$ , possesses a convergent subsequence.

For each  $k$ , let  $S^k$  be the unit sphere of  $\mathbb{R}^{k+1}$  and let us denote by

$$\begin{aligned} \Gamma(S^k, M) &:= \{h \in C(S^k, M) : h \text{ is odd}\}, \\ \gamma_k &:= \inf_{h \in \Gamma(S^{k-1}, M)} \sup_{x \in S^{k-1}} G(h(x)). \end{aligned} \quad (4.1)$$

Using the fact  $b^+ \neq 0$ , we can easily see that  $\Gamma(S^k, M) \neq \emptyset$ . Thus  $\gamma_k$  is finite for all  $k$ . The lemma below was proved by Cuesta (Proposition 2.7, [6]).

**Lemma 4.1** *For each  $k$ , if  $G$  satisfies  $(PS)_{\gamma_k, M}$  then there exists  $u \in M$  such that  $G(u) = \gamma_k$  and  $TG(u) = 0$ .*

Using Lemma 4.1 we obtain the following.

**Theorem 4.1** *The sequence  $\{\gamma_k\}$  given by (4.1) are positive eigenvalues of (1.1).*

*Proof.* If  $u$  is a critical point of  $G$  with respect to  $M$ , then  $u$  solves  $G'(u) = \gamma F'(u)$ , with  $G(u) = \gamma$ . Thus by (2.6),  $(u, \gamma)$  is an eigenpair of (1.1). By Lemma 4.1 it suffices to verify that  $G$  satisfies  $(PS)_{\gamma_k, M}$  for any  $k$ . To see this, we use similar ideas made in the proofs of Lemma 4.31, Zeidler [19] and Theorem 5.3, Szulkin [16]. Given  $k$ , let  $\{u_n\}$  be a sequence in  $M$  such that  $\|TG(u_n)\| \rightarrow 0$  and  $G(u_n) \rightarrow \gamma_k$ . We define for each  $u \in M$  the projection mapping  $P_u : X \rightarrow T_u M$  such that

$$P_u v = v - \frac{\langle F'(u), v \rangle}{\|F'(u)\|^2} J^{-1} F'(u),$$

where  $J : X \rightarrow X^*$  is the duality mapping (see e.g. Section 4, Browder [4] or Proposition 47.19, Zeidler [19]). We recall that  $J$  and  $J^{-1}$  are continuous,  $\|Ju\| = \|u\|$  and  $\langle Ju, u \rangle = \|u\|^2$ ,  $\forall u \in X$ . Thus for any  $v \in X$ ,

$$\langle G'(u), P_u v \rangle = \langle TG(u), P_u v \rangle \leq \|TG(u)\| \|P_u v\| \leq 2\|TG(u)\| \|v\|.$$

In the last inequality we use the fact that  $\|J^{-1}F'(u)\| = \|F'(u)\|$ . Consequently,

$$\sup_{\|v\|\leq 1} \langle G'(u_n), P_{u_n} v \rangle \leq \sup_{\|v\|\leq 1} 2\|TG(u_n)\| \|v\| \rightarrow 0.$$

Hence

$$G'(u_n) - \frac{\langle G'(u_n), J^{-1}F'(u_n) \rangle}{\|F'(u_n)\|^2} F'(u_n) \rightarrow 0. \quad (4.2)$$

As  $G(\cdot) = \|\cdot\|^p$ , the sequence  $\{u_n\}$  is bounded and thus (via a subsequence)  $u_n \rightharpoonup u$  for some  $u$ . It follows from Proposition 3.1 that  $F'(u_n) \rightarrow F'(u)$  and  $u \in M$ . It is easy to see that  $G'(u_n)$  is bounded and  $F'(u_n)$  is bounded away from zero. Therefore, after passing to subsequences, we conclude that  $G'(u_n)$  converges to some  $v$  in  $X^*$ . Condition (H3) in Section 3 and Proposition 3.2 imply that  $u_n \rightarrow u$ .

**Remark 4.1** The characterization (4.1) was first introduced by Drábek and Robinson [8] for the Dirichlet problem (1.5), i.e.  $a \equiv 1$  and  $b \equiv 1$ . We note that in their proof the  $(PS)_{c,M}$  condition is required for the functional

$$I(u) = \frac{G(u)}{F(u)}$$

defined on the manifold  $M$ . For problem (1.1), this fact can be shown by adapting the proof of Theorem 4.1.

## 5 Comparison between $\{\lambda_k\}$ and $\{\gamma_k\}$

The existence of  $\{\lambda_n\}$  in Corollary 3.1 can be obtained using the Ljusternik-Schnirelman critical point theory on  $C^1$  manifolds proved by Szulkin [16]. For each  $n$ , we define

$$\Lambda_k := \{A : A \text{ is a compact, symmetric subset of } M \text{ such that } g(A) \geq k\},$$

where  $M = F^{-1}\{1\}$  and  $g(K)$  is defined as in (3.2). In [16] (Corollary 4.1) Szulkin proved the following.

**Theorem 5.1** *Suppose  $M$  is a closed symmetric  $C^1$  manifold of a real Banach space  $X$  and  $0 \notin M$ . Suppose  $G \in C^1(M, \mathbb{R})$  is even and bounded below. Define*

$$c_k = \inf_{A \in \Lambda_k} \sup_{u \in A} G(u). \quad (5.1)$$

*If  $\Lambda_k \neq \emptyset$  for some  $k \geq 1$  and if  $G$  satisfies  $(PS)_{c,M}$  of all  $c = c_j$ ,  $j = 1, \dots, k$ , then  $F$  has at least  $k$  distinct pairs of critical points.*

We notice that to apply Theorem 5.1 with  $X = W_0^{1,p}(a, \Omega)$  and  $F$  and  $G$  given by (3.5) and (3.6) it is crucial to verify  $G$  satisfies  $(PS)_c$  (see Theorem 4.1).

Next we give a comparison between  $\lambda_k$  and  $\gamma_k$ . A simple calculation shows that

$$\inf_{A \in \Lambda_k} \sup_{u \in A} G(u) = \frac{1}{\sup_{K \in \Sigma_k} \inf_{u \in K} F(u)}.$$

It follows from (3.3), Corollary 3.1 and (5.1) that

$$c_k = \frac{1}{\beta_k} = \lambda_k.$$

Moreover, the set of images  $\{h(S^{k-1}) : h \in \Gamma(S^{k-1}, M)\}$  is a subset of  $\Lambda_k$ . Therefore

$$\lambda_k \leq \gamma_k.$$

**Remark 5.1** It is obvious that

$$\lambda_1 = \gamma_1 = \inf \left\{ \int_{\Omega} a |\nabla u|^p dx : u \in W_0^{1,p}(a, \Omega) \text{ and } \int_{\Omega} b |u|^p dx = 1 \right\}.$$

We refer to [5] and [7] for the properties of the principal eigenvalue  $\lambda_1$ . When  $a \equiv 1$  and  $b$  satisfies (1.4), it was proved in [3] that

$$\lambda_2 = \gamma_2 = \underline{\lambda}_2 := \min\{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue and } \lambda > \lambda_1\}$$

and

$$\underline{\lambda}_2 = \inf_{h \in U} \max_{u \in h([-1,1])} \int_{\Omega} |\nabla u|^p dx,$$

where  $U := \{h \in C([-1,1], M) : h(\pm 1) = \pm \varphi_1\}$  and  $\varphi_1 \in M$  is the positive eigenfunction associated with  $\lambda_1$ .

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