Lagrange Multipliers for Variational Inequalites

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Introduction

- Consider the eigenvalue inequality:

$$\langle Au - \lambda Bu, v - u \rangle \geq 0, \qquad \forall v \in K \tag{1}$$

  where A and B are two possibly nonlinear operators in some function space X. The set K is convex in X.

- If $K \equiv X$, (1) becomes the equation :

$$A(u) - \lambda B(u) = 0 \tag{2}$$

The most important result concerning the eigensolution is the Lagrange multiplier rule:

"Let X be a Banach space and $F, G : X \to \mathbb{R}$ be two Fréchet differential real functionals. If F achieves its minimum restricted to the set $M = \{v \in X : G(v) = G(u)\}$ at the point u and if $G'(u) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that $F'(u) = \lambda G'(u)$ "

- We now consider the more general unconstrained problem given by the inequality:

$$\langle Au - \lambda Bu, v - u\rangle + \psi(v) - \psi(u) \geq 0, \qquad \forall v \in X \quad (3)$$

  where $\psi : X \to \mathbb{R} \cup \infty$ is an extended real proper and convex functional.

- It is clear that (3) reduces to (1) when ψ is the indicator of the convex set K and (3) becomes (2) if $\psi \equiv 0$.

The main result

Let F, G be two Fréchet differential functionals on a normed linear space X with their Fréchet derivatives $A, B : X \to X^*$. Let $\psi : X \to \mathbb{R} \cup \infty$ be a proper, convex, extended real functional. Assume that :

(i) $F + \psi$ achieves its minimum on a non-empty set $M = \{v \in X : G(v) = r\}$ for some real number r at $u \in D(\psi)$, where $D(\psi)$ is the effective domain of ψ.

(ii) There are $h, k \in D(\psi)$ such that $\langle Bu, h - u\rangle > 0$ and $\langle Bu, k - u\rangle < 0$.

Then there exists at least one $\lambda \in \mathbb{R}$ such that :

$$\langle Au - \lambda Bu, v - u\rangle + \psi(v) - \psi(u) \geq 0, \qquad \forall v \in X \tag{4}$$

The proof

It suffices to prove (4) for any $v \in D(\psi)$. From that observation we define:

$$\begin{aligned} \Omega^{+} &= \{v \in D(\psi) : \langle Bu, v-u \rangle > 0\} \\ \Omega^{-} &= \{v \in D(\psi) : \langle Bu, v-u \rangle < 0\} \\ \Omega^{0} &= \{v \in D(\psi) : \langle Bu, v-u \rangle = 0\} \end{aligned}$$

Then $D(\psi) = \Omega^{+} \cup \Omega^{-} \cup \Omega^{0}$.
And the assumption $\Omega^{+} \neq \emptyset$ and $\Omega^{-} \neq \emptyset$.

Now let $v \in \Omega^+$ and $w \in \Omega^-$.
Given $\epsilon > 0$, we define for $t, \alpha, \beta \geq 0$

$$f(t,\alpha,\beta) = u + t(v-u) + tb(w-u) + \alpha(v-u) + \beta(w-u),$$

where $b = -\dfrac{\langle Bu, v-u\rangle}{\langle Bu, w-u\rangle} > 0$.

$$\begin{aligned} G(f(t,\alpha,\beta)) &= G(u) + t\langle Bu, v-u\rangle + tb\langle Bu, w-u\rangle \\ &\quad +\alpha\langle Bu, v-u\rangle + \beta\langle Bu, w-u\rangle \\ &\quad +o(|t| + |\alpha| + |\beta|) \\ &= G(u) + \alpha\langle Bu, v-u\rangle + \beta\langle Bu, w-u\rangle \\ &\quad +o(|t| + |\alpha| + |\beta|). \end{aligned}$$

Let $\alpha, \beta \in [0, \epsilon t]$ then $o(|t| + |\alpha| + |\beta|) = o(|t|)$.
For $t > 0$ sufficiently small enough we obtain:

$$\begin{cases} \text{if } \alpha = \epsilon t, \beta = 0 : G(f) > G(u) \\ \text{if } \alpha = 0, \beta = \epsilon t : G(f) < G(u). \end{cases}$$

Thus there exist $\alpha, \beta \in [0, \epsilon t]$ such that $f(t,\alpha,\beta) \in M$

$$
\begin{aligned}
0 &\leq F(f)+\psi(f)-F(u)-\psi(u) \\
&= F(f)-F(u) \\
&\quad +\psi\left((1-t-tb-\alpha-\beta)u+(t+\alpha)v+(tb+\beta)w\right)-\psi(u) \\
&\leq \langle Au, t(v-u)+tb(w-u)+\alpha(v_u)+\beta(w-u)\rangle+o(|t|) \\
&\quad +(1-t-tb-\alpha-\beta)\psi(u)+(t+\alpha)\psi(v)+(tb+\beta)\psi(w)-\psi(u) \\
&= t\left[\langle Au, v-u\rangle+\psi(v)-\psi(u)+b\left(\langle Au, w-u\rangle+\psi(w)-\psi(u)\right)\right] \\
&\quad +\alpha\langle Au, v-u\rangle+\beta\langle Au, w-u\rangle-(\alpha+\beta)\psi(u)+\alpha\psi(v)+\beta\psi(w)+o(|t|) \\
&\leq tc+3(\alpha+\beta)M+o(|t|),
\end{aligned}
$$

where

$$
\begin{aligned}
c &= \langle Au, v-u\rangle+\psi(v)-\psi(u)+b\left(\langle Au, w-u\rangle+\psi(w)-\psi(u)\right) \\
&= \langle Bu, v-u\rangle\left[\frac{\langle Au, v-u\rangle+\psi(v)-\psi(u)}{\langle Bu, v-u\rangle}-\frac{\langle Au, w-u\rangle+\psi(w)-\psi(u)}{\langle Bu, w-u\rangle}\right]
\end{aligned}
$$

and $M=\max\{|\psi(u)|, |\psi(v)|, |\psi(w)|, |\langle Au, v-u\rangle|, |\langle Au, w-u\rangle|\}$.
Since $\alpha, \beta \in [0, \epsilon t]$,

$$0 \leq tc+6\epsilon Mt+o(|t|).$$

Dividing the above inequality by t and then let $t \to 0^+$ we obtain:

$$0 \leq c+6\epsilon M.$$

Since $\epsilon > 0$ is chosen arbitrarily, it follows that $c \geq 0$. Hence, for any $v \in \Omega^+$ and $w \in \Omega^-$ we have shown that:

$$\frac{\langle Au, v-u\rangle + \psi(v) - \psi(u)}{\langle Bu, v-u\rangle} - \frac{\langle Au, w-u\rangle + \psi(w) - \psi(u)}{\langle Bu, w-u\rangle} \geq 0$$

Therefore,

$$\lambda_2 = \inf_{v\in\Omega^+} \frac{\langle Au, v-u\rangle + \psi(v) - \psi(u)}{\langle Bu, v-u\rangle} \geq \sup_{w\in\Omega^-} \frac{\langle Au, w-u\rangle + \psi(w) - \psi(u)}{\langle Bu, w-u\rangle} = \lambda_1.$$

Let λ be in $[\lambda_1, \lambda_2]$, it is easy to verify that

$$\langle Au - \lambda Bu, v-u\rangle + \psi(v) - \psi(u) \geq 0, \qquad \forall v \in \Omega^+ \cup \Omega^-. \tag{5}$$

If $v \in \Omega^0$, define a sequence $v_n = \frac{1}{n}h + \left(1 - \frac{1}{n}\right)v$. Since $D(\psi)$ is convex and $\langle Bu, v_n - u\rangle = \frac{1}{n}\langle Bu, h - u\rangle > 0$, $v_n \in \Omega^+$. By (5),

$$\begin{aligned} 0 &\leq \langle Au - \lambda Bu, v_n - u\rangle + \psi(v_n) - \psi(u) \\ &\leq \langle Au - \lambda Bu, v_n - u\rangle + \frac{1}{n}\psi(h) + (1 - \frac{1}{n})\psi(v) - \psi(u). \end{aligned}$$

Let $n \to \infty$ we obtain

$$\langle Au - \lambda Bu, v - u\rangle + \psi(v) - \psi(u) \geq 0, \qquad \forall v \in \Omega^0. \quad (6)$$

Combining (5), (6) we have the theorem for any $\lambda \in [\lambda_1, \lambda_2]$.

Remarks

For applications see

- **M.GARCÍA-HUIBOBRO, V.K.LE, R.MANÁSEVICH AND K.SCHMITT**, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting*. NoDEA Nonlinear Differential Equations Appl., 6 (1999), pp. 207-225.

- **J.-P GOSSEZ AND R.MANÁSEVICH**, *On a nonlinear eigenvalue problem in Orlicz-Sobolev spaces*, Proc. Roy. Soc Edingburg Sect. A, 132 (2002), pp 891-909.

- **R.S.KUBRUSLY**, Variational methods for nonlinear eigenvalue inequalities subject to constraints, Trans. Amer. Math. Soc., 347 (1995), pp. 4485-4513.