

VARIATIONAL METHODS FOR NONLINEAR EIGENVALUE INEQUALITIES

AN LE

Department of Mathematics, University of Utah
Salt Lake City, UT 84112, USA

VY LE

Department of Mathematics and Statistics,
Univeristy of Missouri-Rolla
Rola, MO 65409, USA

KLAUS SCHMITT

Department of Mathematics, University of Utah
Salt Lake City, UT 84112, USA

September 4, 2003

Abstract

We prove the existence of solutions for nonlinear eigenvalue inequality.

1 Introduction

- Consider the eigenvalue inequality:

$$\langle Au - \lambda Bu, v - u \rangle \geq 0, \quad \forall v \in K \quad (1)$$

where A and B are two possibly nonlinear operators in some function space X . The set K is convex in X .

- If $K \equiv X$, (1) becomes the equation :

$$A(u) - \lambda B(u) = 0 \quad (2)$$

The most important result concerning the eigensolution is the Lagrange multiplier rule: "Let X be a Banach space and $F, G : X \rightarrow \mathbb{R}$ be two Fréchet differential real functionals. If F achieves its minimum restricted to the set $M = \{v \in X : G(v) = G(u)\}$ at the point u and if $G'(u) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that $F'(u) = \lambda G'(u)$ "

We now consider the more general unconstrained problem given by the inequality:

$$\langle Au - \lambda Bu, v - u \rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X \quad (3)$$

where $\psi : X \rightarrow \mathbb{R} \cup \infty$ is an extended real proper and convex functional.

2 The main result

Theorem 2.1. *Let F, G be two Fréchet differential functionals on a normed linear space X with their Fréchet derivatives $A, B : X \rightarrow X^*$. Let $\psi : X \rightarrow \mathbb{R} \cup \infty$ be a proper, convex, extended real functional. Assume that :*

(i) *$F + \psi$ achieves its minimum on a non-empty set $M = \{v \in X : G(v) = r\}$ for some real number r at $u \in D(\psi)$, where $D(\psi)$ is the effective domain of ψ .*

(ii) *There are $h, k \in D(\psi)$ such that $\langle Bu, h - u \rangle > 0$ and $\langle Bu, k - u \rangle < 0$. Then there exists at least one $\lambda \in \mathbb{R}$ such that :*

$$\langle Au - \lambda Bu, v - u \rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in X \quad (4)$$

Remark 2.2.

- (i) This theorem generalizes the Lagrange multiplier rule. By letting $\psi = 0$ we get $A(u) = \lambda B(u)$.

(ii) Let K be a convex set in X and let ψ be the indicator function of K , it follows from (3) that

$$\langle Au - \lambda Bu, v - u \rangle \geq 0, \quad \forall v \in K$$

(iii) The theorem was first proved by Kubrusly in [4] using an indirect argument and was applied in [2] and [3] to solve the eigenvalue problems in Orlicz-Sobolev spaces.

Proof. It suffices to prove (4) for any $v \in D(\psi)$. From that observation we define:

$$\begin{aligned} \Omega^+ &= \{v \in D(\psi) : \langle Bu, v - u \rangle > 0\} \\ \Omega^- &= \{v \in D(\psi) : \langle Bu, v - u \rangle < 0\} \\ \Omega^0 &= \{v \in D(\psi) : \langle Bu, v - u \rangle = 0\} \end{aligned}$$

Then $D(\psi) = \Omega^+ \cup \Omega^- \cup \Omega^0$.

By the assumption $\Omega^+ \neq \emptyset$ and $\Omega^- \neq \emptyset$. Now let $v \in \Omega^+$ and $w \in \Omega^-$.

Given $\epsilon > 0$, we define for $t, \alpha, \beta \geq 0$

$$f(t, \alpha, \beta) = u + t(v - u) + tb(w - u) + \alpha(v - u) + \beta(w - u),$$

where $b = -\frac{\langle Bu, v - u \rangle}{\langle Bu, w - u \rangle} > 0$.

Then

$$\begin{aligned} G(f(t, \alpha, \beta)) &= G(u) + t\langle Bu, v - u \rangle + tb\langle Bu, w - u \rangle + \alpha\langle Bu, v - u \rangle + \beta\langle Bu, w - u \rangle \\ &\quad + o(|t| + |\alpha| + |\beta|) \\ &= G(u) + \alpha\langle Bu, v - u \rangle + \beta\langle Bu, w - u \rangle + o(|t| + |\alpha| + |\beta|). \end{aligned}$$

Let $\alpha, \beta \in [0, \epsilon t]$ then $o(|t| + |\alpha| + |\beta|) = o(|t|)$.

Furthermore, for $t > 0$ sufficiently small enough we obtain :

$$\begin{cases} \text{if } \alpha = \epsilon t, \beta = 0 : G(f) > G(u) \\ \text{if } \alpha = 0, \beta = \epsilon t : G(f) < G(u). \end{cases}$$

Fix $\beta = 0$, let α vary from ϵt to 0 then either $G(f) - G(u)$ changes sign or $G(f) > G(u)$ for all $0 \leq \alpha \leq \epsilon t$.

For the first case there is $\alpha = \alpha(t) \in [0, \epsilon t]$ such that $G(f) = G(u)$. And

for the second case, we fix $\alpha = 0$ and let β vary from 0 to ϵt , there is $\beta = \beta(t) \in [0, \epsilon t]$ such that $G(f) = G(u)$.

In general, for each $t > 0$ there exist $\alpha, \beta \in [0, \epsilon t]$ such that $f(t, \alpha, \beta) \in M$ and thus for any $t > 0$ sufficiently small enough :

$$\begin{aligned}
0 &\leq F(f) + \psi(f) - F(u) - \psi(u) \\
&= F(f) - F(u) + \psi((1-t-tb-\alpha-\beta)u + (t+\alpha)v + (tb+\beta)w) - \psi(u) \\
&\leq \langle Au, t(v-u) + tb(w-u) + \alpha(v_u) + \beta(w-u) \rangle + o(|t|) \\
&\quad + (1-t-tb-\alpha-\beta)\psi(u) + (t+\alpha)\psi(v) + (tb+\beta)\psi(w) - \psi(u) \\
&= t[\langle Au, v-u \rangle + \psi(v) - \psi(u) + b(\langle Au, w-u \rangle + \psi(w) - \psi(u))] \\
&\quad + \alpha\langle Au, v-u \rangle + \beta\langle Au, w-u \rangle - (\alpha+\beta)\psi(u) + \alpha\psi(v) + \beta\psi(w) + o(|t|) \\
&\leq tc + 3(\alpha+\beta)M + o(|t|),
\end{aligned}$$

where

$$\begin{aligned}
c &= \langle Au, v-u \rangle + \psi(v) - \psi(u) + b(\langle Au, w-u \rangle + \psi(w) - \psi(u)) \\
&= \langle Bu, v-u \rangle \left[\frac{\langle Au, v-u \rangle + \psi(v) - \psi(u)}{\langle Bu, v-u \rangle} - \frac{\langle Au, w-u \rangle + \psi(w) - \psi(u)}{\langle Bu, w-u \rangle} \right],
\end{aligned}$$

and $M = \max\{|\psi(u)|, |\psi(v)|, |\psi(w)|, |\langle Au, v-u \rangle|, |\langle Au, w-u \rangle|\}$.

Since $\alpha, \beta \in [0, \epsilon t]$,

$$0 \leq tc + 6\epsilon Mt + o(|t|).$$

Dividing the above inequality by t and then let $t \rightarrow 0^+$ we obtain:

$$0 \leq c + 6\epsilon M.$$

Since $\epsilon > 0$ is chosen arbitrarily, it follows that $c \geq 0$. Hence, for any $v \in \Omega^+$ and $w \in \Omega^-$ we have shown that:

$$\frac{\langle Au, v-u \rangle + \psi(v) - \psi(u)}{\langle Bu, v-u \rangle} - \frac{\langle Au, w-u \rangle + \psi(w) - \psi(u)}{\langle Bu, w-u \rangle} \geq 0$$

Therefore,

$$\lambda_2 = \inf_{v \in \Omega^+} \frac{\langle Au, v-u \rangle + \psi(v) - \psi(u)}{\langle Bu, v-u \rangle} \geq \sup_{w \in \Omega^-} \frac{\langle Au, w-u \rangle + \psi(w) - \psi(u)}{\langle Bu, w-u \rangle} = \lambda_1.$$

Let λ be in $[\lambda_1, \lambda_2]$, it is easy to verify that

$$\langle Au - \lambda Bu, v-u \rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in \Omega^+ \cup \Omega^-. \quad (5)$$

If $v \in \Omega^0$, define a sequence $v_n = \frac{1}{n}h + \left(1 - \frac{1}{n}\right)v$. Since $D(\psi)$ is convex and $\langle Bu, v_n - u \rangle = \frac{1}{n}\langle Bu, h - u \rangle > 0$, $v_n \in \Omega^+$. By (5),

$$\begin{aligned} 0 &\leq \langle Au - \lambda Bu, v_n - u \rangle + \psi(v_n) - \psi(u) \\ &\leq \langle Au - \lambda Bu, v_n - u \rangle + \frac{1}{n}\psi(h) + \left(1 - \frac{1}{n}\right)\psi(v) - \psi(u). \end{aligned}$$

Let $n \rightarrow \infty$ we obtain

$$\langle Au - \lambda Bu, v - u \rangle + \psi(v) - \psi(u) \geq 0, \quad \forall v \in \Omega^0. \quad (6)$$

Combining (5), (6) we have the theorem for any $\lambda \in [\lambda_1, \lambda_2]$. \square

3 An application to obstacle problems

Let Ω be an open bounded domain in \mathbb{R}^N , $p > 1$ and q be such that the Sobolev the compact imbedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds. Let $f \in W_0^{1,p}(\Omega)$, $f \geq 0$. We will find $(u, \lambda) \in K \times \mathbb{R}$ such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u (\nabla v - \nabla u) - \lambda \int_{\Omega} |u|^{q-2} u (v - u) \geq 0, \forall v \in K, \quad (7)$$

where $K = \{u \in W_0^{1,p}(\Omega) : u \geq f \text{ on } \Omega\}$ is a closed convex set of $W^{1,p}(\Omega)$. Let ψ be the indicator functional of K ; that is

$$\begin{cases} \psi(u) = 0, & \text{if } u \in K \\ \psi(u) = \infty, & \text{otherwise.} \end{cases}$$

Now, define on $W_0^{1,p}(\Omega)$:

$$\begin{aligned} F(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx \\ G(u) &= \frac{1}{q} \int_{\Omega} u^q dx \end{aligned}$$

Then the differential operators A, B of F, G (respectively) are

$$\begin{aligned} \langle Au, v \rangle &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx \\ \langle Bu, v \rangle &= \int_{\Omega} |u|^{q-2} u v dx \end{aligned}$$

Let $r > 0$ be a real number such that $r > G(f)$. By the compact imbedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, G is weakly continuous and thus the level set $G_r = \{u \in W_0^{1,p}(\Omega) : G(u) = r\}$ is weakly closed.

We observe that F is a norm in $W_0^{1,p}(\Omega)$ and ψ is convex lower semi-continuous and positive. Thus $F + \psi$ is weak lower semi-continuous and coercive on $W_0^{1,p}(\Omega)$. Hence $F + \psi$ achieves its minimum on G_r at some $u_r \in G_r$.

Since $(F + \psi)(u) = (F + \psi)(|u|)$, we can assume $u_r \geq 0$. Moreover, by the choosing of r , $u_r \neq f$.

To apply theorem 2.1 it remains to verify (ii). To see this, let $v_1 = f$ and $v_2 = 2u_r$ be in K , then

$$\begin{cases} \int_{\Omega} |u_r|^{q-2} u_r (v_1 - u_r) < 0 \\ \int_{\Omega} |u_r|^{q-2} u_r (v_2 - u_r) > 0 \end{cases}$$

Therefore, there is $\lambda \in \mathbb{R}$ such that (u_r, λ) solves (7).

References

- [1] F. BROWDER, *Variational methods for nonlinear elliptic eigenvalue problems*, Bull. Amer. Math. Soc., 71 (1965), pp. 176–183.
- [2] M. GARCÍA-HUIDOBRO, V. K. LE, R. MANÁSEVICH, AND K. SCHMITT, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting*, NoDEA Nonlinear Differential Equations Appl., 6 (1999), pp. 207–225.
- [3] J.-P. GOSSEZ AND R. MANÁSEVICH, *On a nonlinear eigenvalue problem in Orlicz-Sobolev spaces*, Proc. Roy. Soc. Edinburgh Sect. A, 132 (2002), pp. 891–909.
- [4] R. S. KUBRUSLY, *Variational methods for nonlinear eigenvalue inequalities*, Differential Integral Equations, 3 (1990), pp. 923–932.
- [5] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications, Vol.3: Variational Methods and Optimization*, Springer, Berlin, 1985.