Worksheet #6

Worksheet #6 is due Thursday, October 8th. You are encouraged to work with other people to solve these problems, but you have to write the solutions down individually. Please turn in these sheets only. You can write at the bottom of each page and/or on the back. Show all the steps and explain your reasoning when necessary: correct solutions with no explanation do not count as complete. Partial credit will be assigned to incomplete answers.

Problem 1. (20 points) Remember Batman and Superman? They’re still on their mission to save the world! This time they need our help with some related rates. Let the situation be exactly the same as in Problem 2 from Worksheet #1.

(a) Find the related change of velocity between the two superheroes at noon. [Hint: use implicit differentiation]

(b) Batman has a problem with his car: the Batmobile is a bit rusty, and it could break if the fuel flows too quickly out of its fuel tank. The fuel tank has the shape of a reversed cone of height 4 feet and base radius 2 feet, and using a stick Batman is able to evaluate that the rate of change $\frac{dh}{dt}$ of the level $h$ of the fuel is $\frac{1}{3}$ feet per hour, when $h = 2$ feet. Knowing that the car breaks if the constant rate at which fuel flows ($\frac{dV}{dt}$) is greater than 2 cubed feet per hour, will Batman be safe or not?
Problem 2. (10 points) Given a function $f(x)$ and a point $a$ in its domain, we call the $n$-th Taylor polynomial of $f(x)$ centered at $a$ the following polynomial

$$T_n,a(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x - a)^n.$$ 

This polynomial has the remarkable property of having the first $n$ derivatives at the point $a$ equal to the ones of $f(x)$ (you can check it!). This tool is often used to approximate the value of the function $f(x)$ near the point $a$ (and actually this polynomial itself or an improved version of it is what your calculator evaluates when you ask it the value of a non-polynomial function, like $e^x$).

(a) Be your own calculator, and find the value of $e$ at the 3rd decimal, using a Taylor polynomial for $e^x$ centered at the point $a = 0$, with sufficiently high degree. (Check it on your calculator afterwards!).

Sometimes Taylor polynomials fail at their job. Consider the function

$$f(x) = \begin{cases} 
  e^{-1/x} & \text{for } x > 0 \\
  0 & \text{for } x \leq 0
\end{cases}.$$ 

(b) Prove that the first two derivatives of $f(x)$ at the point $a = 0$ are all 0. This is indeed true for any derivative of $f(x)$, so that $f^{(k)}(0) = 0$ for all $k$ (I am not asking you to prove it, take it for granted). What can you tell about Taylor polynomials centered at 0 for $f(x)$? Can we use them to approximate $f(1) = \frac{1}{e}$?
Problem 3. (Bonus problem, 5 points) Prove that there is no differentiable function $f : [-1, 1] \to \mathbb{R}$ with the following properties

(i) $f(0) = 0$;

(ii) $f(-1) = f(1) = 1$;

(iii) $|f'(x)| \leq 1$ for all $x \in (-1, 1)$.

[Step by step hint: Let $y \in (0, 1), O = (0, 0), P = (1, 1), Q = (y, f(y))$. Suppose $f(y) \neq y$, and look at the slopes of the segments $\overline{OQ}$ and $\overline{QP}$; what can you say about at least one of them? What does the Mean Value Theorem tells you then? Is this possible given (iii)? Conclude $f(x) = x$ for all $x \in (0, 1)$. Apply a similar argument for $x \in (-1, 0)$, and conclude that a function satisfying (i),(ii) and (iii) must be $f(x) = |x|$, which is not differentiable.]