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Victor DEVOUE

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To my Parents

To my Sisters

To my brothers

Introduction

The solution to many differential systems depends upon Cauchy data or initial data. However the Cauchy problem is not well posed in many cases, even hyperbolic ones.

It is the case when the data are too irregular or when they are given on a characteristic manifold.

For example, for a nonlinear differential system, the examination of the data like Dirac measure on a (non characteristic) manifold has no meaning in the distributions theory. Even for a linear problem, if the criterion of Hörmander "Wave Front Set" on the restriction of the distributions to a submanifold is not verified, there is no standard means to formulate correctly the Cauchy problem in a distributional context.

In the characteristic case, even if the data are analytical, the Cauchy-Kowaleska theorem cannot be used. This theorem proposes a construction of the solution u, at least locally, from data regular enough carried by a manifold on which we can first calculate formally all the partial derivatives of u. Such a manifold is said to be non characteristic and we can, in the linear case, characterize it, simply enough by an equation and even a system.

The nonlinear case, "non characteristic", is more difficult to formulate for the higher orders and systems; it is not certain that there is a consensus for a general definition relative to the nonlinear differential systems of a non definite order.

In the characteristic case, the formal calculus of the partial derivatives on the manifold carrying the data meet with a geometric obstruction which is difficult to get around. For linear characteristic problems, some results of existence, but not of uniqueness, are proved in the field of the distributions in a half-space ([4], [6]). Other results are proved (Garding, Kotake, Leray, Wagschal, Hamada, Dunau) in complex framework (where the solutions may be holomorphic and may have ramifications around characteristic curves issued from characteristics). However, we do not know any general answer in real anlytical or C^∞ cases and for nonlinear problems. For these cases, and also for linear cases, the characteristic problems are those we "fall into the holes" of the canonical stratification defined in the Shih Weishu theory [18], [19] and it is proved by Shi Wei Hui [17] that the Cauchy problem is not well posed for the Navier-Stokes equations, on the hyperplane {t=0}.

The purpose of this work is to propose a method to solve some Cauchy problems with irregular or characteristic data by using the recent theories of generalized functions [1], [3], [15] and particularly the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras [8], [11] defined in the research works of the Non-linear Algebraic Analysis group.

- We search a generalized solution u, with a meaning that we will be defined later, to the following Cauchy problem:

$$(P)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|\gamma} = \varphi \\ \dfrac{\partial u}{\partial y}_{|\gamma} = \psi \end{cases}$$

where φ and ψ are one-variable generalized functions. The initial values are given on the monotonous γ curve of equation $y = f(x)$. The notation $F(.,.,u)$ extends, with a meaning that we will be defined later, the expression:

$$(x, y) \mapsto F(x, y, u(x, y))$$

in the case where u is a generalized function of two variables x and y.

We study the case where the data are carried on a characteristic curve $\gamma = (Ox)$ from the previous results.

- We search a generalized solution u, with a meaning that we will be defined later, of the following Goursat problem:

$$(P')\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u=F(.,.,u)\\[2ex] u_{|(Ox)}\ =\varphi\\[2ex] u_{|\gamma}\ =\psi\end{cases}$$

where φ and ψ are generalized functions of one real variable. The initial values are given on a characteristic curve C: (Ox), and on a monotonous curve γ of equation $x=g\,(y)$. The notation $F(.,.,u)$ extends, with a meaning that we will be defined later, the expression:

$$(x,y)\rightarrow F(x,y,u(x,y))$$

in the case where u is a generalized function of two variables x and y.

- In the **first part** we study primarily the regular problems.

We solve the following regular Cauchy problem:

$$(P)\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u=F(.,.,u)\\[2ex] u_{|\gamma}\ =\varphi\\[2ex] \dfrac{\partial u}{\partial y}_{|\gamma}\ =\psi\end{cases}$$

where φ and ψ are smooth functions of one real variable. The initial values are given on the monotonous curve γ of equation $y=f(x)$ and the mapping $(x,y)\mapsto F(x,y,u(x,y))$ is a smooth function of its arguments.

We solve the following Goursat problem:

$$(P')\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u=F(.,.,u)\\[2ex] u_{|(Ox)}\ =\varphi\\[2ex] u_{|\gamma}\ =\psi\end{cases}$$

where φ and ψ are one-variable smooth functions. The initial values are given on a characteristic curve C: (Ox), and on a monotonous curve γ of equation $x=g\,(y)$ and the mapping $(x,y)\rightarrow F(x,y,u(x,y))$ is a smooth function of its arguments.

- The **second part** is devoted to the definition of the algebras of generalized functions and to the setting of an algebra of generalized functions adapted to the generalized Cauchy problem. Here is an idea of the construction of these algebras:

$\mathbb{K}$ is the field of reals or of complexes and Λ is a set of indexes. $\mathcal{C}$ is the factor ring A/I where I is an ideal of A, a subring of ring $\mathbb{K}^{\Lambda}$.

$(\mathcal{E},\mathcal{P})$ is a sheaf of $\mathbb{K}$-topological algebras on a topological space X. A sheaf of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras on X is a factor sheaf $\mathcal{A}=\mathcal{H}/\mathcal{J}$ where $\mathcal{J}$ is an ideal of $\mathcal{H}$, subsheaf of $\mathcal{E}^{\Lambda}$. The sections of $\mathcal{H}$ (resp $\mathcal{J}$) must verify some estimations in which $\mathcal{H}$ and A (resp I) play a part.

In such algebras, we have good tools to pose and solve many nonlinear differential problems with n irregular data [12], [14].

We choose $\mathcal{E}=\mathrm{C}^{\infty}$, $X=\mathbb{R}^{d}$ for $d=1,2$, $E=\mathcal{D}'$ and $\Lambda=]0,1]$. For every Ω, open subset of $\mathbb{R}^{d}$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$-topology of the uniform convergence of all the derivatives on the compact subspaces of Ω. This topology may be defined by the family of semi-norms $P_{K,l}(u_{\varepsilon})=\sup\limits_{|\alpha|\leq l}\left(\sup\limits_{x\in K}|D^{\alpha}u_{\varepsilon}(x)|\right)$ with $K\Subset\Omega$ and $D^{\alpha}=\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial z_1^{\alpha_1}\ldots\partial z_d^{\alpha_d}}$ for $z=(z_1,\ldots,z_d)\in\Omega$, $l\in\mathbb{N}$ and $\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{N}^{d}$.

Let A a subring of the ring $\mathbb{R}^{\Lambda}$ of the families of reals endowed with the usual operations, and I_A an ideal of A such that A and I_A are stable under overestimation. We suppose that $(1)_{\varepsilon}\in A$.

To simplify, we will put $\mathcal{X}=\mathcal{H}_{(A,\mathrm{C}^{\infty},\mathcal{P})}$, $\mathcal{N}=\mathcal{J}_{(I_A,\mathrm{C}^{\infty},\mathcal{P})}$ et $\mathcal{A}=\mathcal{X}/\mathcal{N}$. We put:

$$\mathcal{X}(\Omega)=\left\{(u_{\varepsilon})_{\varepsilon}\in[\mathrm{C}^{\infty}(\Omega)]^{\Lambda}:\forall K\Subset\Omega,\forall l\in\mathbb{N},\ \left(P_{K,l}(u_{\varepsilon})\right)_{\varepsilon}\in A_{+}\right\}$$

$$\mathcal{N}(\Omega)=\left\{(u_{\varepsilon})_{\varepsilon}\in[\mathrm{C}^{\infty}(\Omega)]^{\Lambda}:\forall K\Subset\Omega,\forall l\in\mathbb{N},\ \left(P_{K,l}(u_{\varepsilon})\right)_{\varepsilon}\in I_{A}^{+}\right\}.$$

The ring of generalized constants associated to the factor algebra is nothing else than the factor ring $\mathcal{C}=A/I_A$.

We introduce the notion of algebra $\mathcal{A}\left(\mathbb{R}^{2}\right)$ stable under F (relatively to $\mathcal{C}=A/I_A$.).

Ω being an open subset of $\mathbb{R}^d$, for $x \in \Omega$ and $u = [u_\varepsilon] \in \mathcal{A}(\Omega)$, we define the $\mathcal{D}'$-parametric singular spectrum of $u \in \mathcal{A}(\Omega)$ as the subset of $\Omega \times \mathbb{R}_+$: $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u = \big\{(x,r) \in \Omega \times \mathbb{R}_+, r \in \Sigma_{\mathcal{D}',x}(u)\big\}$ where $\Sigma_{\mathcal{D}',x}(u) = \mathbb{R}_+ \backslash N_{\mathcal{D}',x}(u)$ is the $\mathcal{D}'$-fiber over x and $N_{\mathcal{D}',x}(u) = \Big\{r \in \mathbb{R}_+; \exists V_x \in \mathcal{V}(x) : \lim\limits_{\varepsilon \to 0}(\varepsilon^r u_{\varepsilon|V_x}) \in \mathcal{D}'(V_x)$

- These tools allow us to tackle the generalized problems in the **third part**.

We search a solution u, in $\mathcal{A}\left(\mathbb{R}^2\right)$, to the generalized Cauchy problem:

$$(P_G)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|\gamma} = \varphi \\ \dfrac{\partial u}{\partial y}_{|\gamma} = \psi. \end{cases}$$

After giving a meaning to (P_G), we show that, if $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under F, if $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}\left(\mathbb{R}^2\right)$ are built on the same ring $\mathcal{C} = A/I$ of generalized constants and, if the data of problem (P_G) verify the conditions $\varphi \in \mathcal{A}(\mathbb{R})$, $\psi \in \mathcal{A}(\mathbb{R})$, $f \in \mathrm{C}^\infty(\mathbb{R})$, then problem (P_G) admits a unique solution u in $\mathcal{A}\left(\mathbb{R}^2\right)$.

Then we make a qualitative study of the solution, notably for $F = 0$ and $f(x) = ax$.

We can study a generalized Goursat problem in the same way:

$$(P_G)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|(Ox)} = \varphi \\ u_{|\gamma} = \psi. \end{cases}$$

- We can then deal with the characteristic problems in the **fourth part.**

We intend to extend some results [10] to general cases, by approaching some characteristic problems by some families of non-characteristic problems and by interpreting the results algebraically.

We study the case where the data are situated on the characteristic curve $\gamma = (Ox)$ from the above results.

The characteristic irregular Cauchy problem:

$$(P_C)\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u = F(.,.,u)\\ u_{|(Ox)} \ = \varphi\\ \dfrac{\partial u}{\partial y}_{|(Ox)} = \psi\end{cases}$$

has no smooth solution (not even C^2) even if data φ and ψ are smooth too. We replace it by the family of non-characteristic problems $(P_\varepsilon)_\varepsilon$:

$$(P_\varepsilon)\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u = F(.,.,u)\\ u_{|_{\gamma_\varepsilon}} = \varphi\\ \dfrac{\partial u}{\partial y}_{|_{\gamma_\varepsilon}} = \psi\end{cases}$$

by considering the curve γ_ε of equation $y = \varepsilon x$ as data. We also try to give a meaning to the family of solutions by putting it in terms of generalized functions belonging to an appropriately defined algebra.

We study the case where the data are regular.

If u_ε is the solution to problem (P_ε), the family $(u_\varepsilon)_\varepsilon$ is the representative of a generalized function which belongs to algebra $\mathcal{A}\left(\mathbb{R}^2\right)$. Then $u = [u_\epsilon]$ is a generalized function that we consider as the generalized solution to the characteristic Cauchy problem (P_C).

Then we study the case of irregular data.

We also give a meaning to the characteristic Cauchy problem (P_C) in the case where φ and ψ are themselves irregular data, (for example some generalized functions), by replacing it by the family of non-characteristic problems $\left(P_{(\varepsilon,\eta)}\right)_{(\varepsilon,\eta)}$ in an appropriate algebra:

$$P_{(\varepsilon,\eta)}\begin{cases}\dfrac{\partial^2 u_{(\varepsilon,\eta)}}{\partial x\partial y}(x,y) = F(x,y,u_{(\varepsilon,\eta)}(x,y))\\ u_{(\varepsilon,\eta)}\,(x,\varepsilon x) = \varphi_\eta(x)\\ \dfrac{\partial u_{(\varepsilon,\eta)}}{\partial y}\,(x,\varepsilon x) = \psi_\eta(x)\end{cases}$$

where $(\varphi_\eta)_\eta$ and $(\psi_\eta)_\eta$ are representatives of φ and ψ.

The parameter ε then boils down to a non-characteristic problem that parameter η makes regular.

If $u_{(\varepsilon,\eta)}$ is the solution to problem $P_{(\varepsilon,\eta)}$, the family $(u_{(\varepsilon,\eta)})_{(\varepsilon,\eta)}$ is representative of a generalized function. So $u = \left[u_{(\varepsilon,\eta)}\right]$ is a generalized function that we consider as the generalized solution to the characteristic Cauchy problem (P_C).

We apply these results to particular cases where $F = 0$.

Part I

Regular problems

Chapter 1

A Cauchy problem with regular data

1.1 Terms of the problem

1.1.1 Two formulations (P_∞) and (P_{int})

We search for a solution u to the following Cauchy problem:

$$(P)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|\gamma} = \varphi \\ \dfrac{\partial u}{\partial y} \mid_\gamma = \psi \end{cases}$$

where f, φ, ψ: $\mathbb{R} \mapsto \mathbb{R}$ are some smooth one-variable functions, γ is the curve whose equation is $y = f(x)$ and $F : (x, y) \mapsto F(x, y, u(x, y))$ is a function smooth of its arguments.

In all cases the following hypothesis will be satisfied:

$$(H):\begin{cases} F \in \mathrm{C}^{\infty}(\mathbb{R}^3, \mathbb{R}) \\ \forall K \Subset \mathbb{R}^2, \quad \sup\limits_{(x,y)\in K; z\in\mathbb{R}} |\partial_z F(x,y,z)| < +\infty \\ f \text{ is defined and strictly increasing on } \mathbb{R}, \text{ whose image is } \mathbb{R} \\ \forall x \in \mathbb{R}, f'(x) \neq 0, \end{cases}$$

where the notation $K \Subset \mathbb{R}^2$ means that K is a compact of $\mathbb{R}^2$.

We denote by (P_∞) the problem which consits in searching for a function $u \in \mathrm{C}^2(\mathbb{R}^2)$ verifying:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y)) & (1) \\ u\left(x, f(x)\right) = \varphi(x) & (2) \\ \dfrac{\partial u}{\partial y}\left(x, f(x)\right) = \psi(x). & (3) \end{cases}$$

We denote by (P_{int}) the problem which consits in searching for a function $u \in \mathrm{C}^0(\mathbb{R}^2)$ verifying:

$$u(x,y) = u_0(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta \qquad (4)$$

where:

$$u_0(x,y) = \chi\left(y\right) - \chi(f(x)) + \varphi(x)$$

and χ denotes a primitive of $\psi \circ f^{-1}$, with:

$$D(x,y,f) = \begin{cases} \left\{(\xi,\eta),\ f^{-1}(y) \leq \xi \leq x;\ y \leq \eta \leq f(\xi)\right\} & \text{if } y \leq f(x) \\ \left\{(\xi,\eta), x \leq \xi \leq f^{-1}(y);\ f(\xi) \leq \eta \leq y\right\} & \text{if } y \geq f(x). \end{cases}$$

1.1.2 Equivalence of the two formulations (P_∞) and (P_{int})

1.1.2.1. Theorem.

Let $u \in \mathrm{C}^0(\mathbb{R}^2)$.

The function u *is solution to* (P_∞) *if and only if* u *is solution to* (P_{int}).

Proof.

Necessary condition.

The existence of f^{-1} is assured by (H). We consider the points $M(x,y)$, $P(f^{-1}(y),y)$, $Q(x,f(x))$, the hypothesis (H) assure that the domain $D(x,y,f)$, defined in 1.1.1., limited by the "curvilinear triangle MPQ " is bounded.

If u is solution to (P_∞), let us suppose that: $y \geq f(x)$.

$$\begin{aligned}
\iint_{D(x,y,f)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta) d\xi d\eta &= \int_{f(x)}^{y} \left(\int_{x}^{f^{-1}(\eta)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta) d\xi \right) d\eta \\
&= \int_{f(x)}^{y} \left[\frac{\partial u}{\partial y}(\xi,\eta) \right]_{\xi = x}^{\xi = f^{-1}(\eta)} d\eta \\
&= \int_{f(x)}^{y} \frac{\partial u}{\partial y}(f^{-1}(\eta),\eta) d\eta - \int_{f(x)}^{y} \frac{\partial u}{\partial y}(x,\eta) d\eta \\
&= \int_{f(x)}^{y} \psi(f^{-1}(\eta)) d\eta - [u(x,\eta)]_{f(x)}^{y} \\
&= \chi\,(y) - \chi(f(x)) - u(x,y) + u\,(x, f(x)) \\
&= \chi\,(y) - \chi(f(x)) - u(x,y) + \varphi(x),
\end{aligned}$$

where χ denotes a primitive of $\psi \circ f^{-1}$.

We have then:

$$\begin{aligned}
u(x,y) &= \chi\,(y) - \chi(f(x)) + \varphi(x) - \iint_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta)) d\xi d\eta \\
&= u_0(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta)) d\xi d\eta,
\end{aligned}$$

where $u_0(x,y) = \chi\,(y) - \chi(f(x)) + \varphi(x)$.

We obtain the same result if we suppose $y \leq f(x)$.

Then u verifies (P_{int}).

Sufficient condition.

If u verifies (P_{int}), suppose that we have $y \geq f(x)$, we can write:

$$\begin{aligned}u(x,y) &= u_0(x,y) - \int_x^{f^{-1}(y)} \left(\int_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))d\eta \right) d\xi \\ &= u_0(x,y) - \int_x^{f^{-1}(y)} G(\xi,y)d\xi,\end{aligned}$$

where: $G(\xi,y) = \int_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))d\eta$.

As we have: $u \in \mathrm{C}^0(\mathbb{R}^2)$, G is a continuous function of ξ and y, hence:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u_0}{\partial x}(x,y) + G(x,y) = \frac{\partial u_0}{\partial x}(x,y) + \int_{f(x)}^{y} F(x,\eta,u(x,\eta))d\eta$$

and consequently:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y) = \frac{\partial^2 u_0}{\partial y \partial x}(x,y) + F(x,y,u(x,y))$$

and as:

$$\frac{\partial^2 u_0}{\partial y \partial x}(x,y) = 0,$$

we have:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y) = F(x,y,u(x,y)).$$

Let us calculate again $u(x,y)$ in the following way:

$$\begin{aligned}u(x,y) &= u_0(x,y) - \int_{f(x)}^{y} \left(\int_x^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))d\xi \right) d\eta \\ &= u_0(x,y) - \int_{f(x)}^{y} H\left(x,\eta\right) d\eta,\end{aligned}$$

where: $H\left(x,\eta\right) = \int_x^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))d\xi$.

As we have $u \in \mathrm{C}^0(\mathbb{R}^2)$, H is a continuous function of x and η, hence:

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u_0}{\partial y}(x,y) - \int_x^{f^{-1}(y)} F(\xi,y,u(\xi,y))d\xi$$

then:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)(x,y) = \frac{\partial^2 u_0}{\partial x \partial y}(x,y) + F(x,y,u(x,y)) = F(x,y,u(x,y)).$$

Finally, we can reverse the order of the partial derivatives and:

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y)).$$

Furthermore:

$$u\left(x,f(x)\right) = u_0\left(x,f(x)\right) = \varphi(x),$$
$$\frac{\partial u}{\partial y}\left(x,f(x)\right) = \frac{\partial u_0}{\partial y}\left(x,f(x)\right) = \psi\circ f^{-1}\left(f(x)\right) = \psi\left(x\right).$$

- This results are unchanged if we suppose: $y \leq f(x)$, so u verifies (P_∞).

If u is of class C^1 then $(x,y) \mapsto F(x,y,u(x,y))$ is of class C^1, then:

$$W : (x,y) \mapsto u_0(x,y) - \int_x^{f^{-1}(y)} \left(\int_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))d\eta\right) d\xi = u_0(x,y) - \int_x^{f^{-1}(y)} G(\xi,y)d\xi$$

have a partial derivative in x of class C^1, and:

$$W : (x,y) \mapsto u_0(x,y) - \int_{f(x)}^{y} \left(\int_x^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))d\xi\right) d\eta = u_0(x,y) - \int_{f(x)}^{y} H\left(x,\eta\right) d\eta$$

have a partial derivative in y of class C^1. As:

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x,y) = F(x,y,u(x,y)) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x,y)$$

is of class C^1 then $u = W$ is of class C^2.

We remark moreover that if u is of class C^n then $(x,y) \mapsto F(x,y,u(x,y))$ is of class C^n, therefore:

$$W : (x,y) \mapsto u_0(x,y) - \int_x^{f^{-1}(y)} \left(\int_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))d\eta\right) d\xi = u_0(x,y) - \int_x^{f^{-1}(y)} G(\xi,y)d\xi$$

have a partial derivative in x of class C^n, and:

$$W : (x,y) \mapsto u_0(x,y) - \int_{f(x)}^{y} \left(\int_x^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))d\xi\right) d\eta = u_0(x,y) - \int_{f(x)}^{y} H\left(x,\eta\right) d\eta$$

have a partial derivative in y of class C^n. As:

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x,y) = F(x,y,u(x,y)) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x,y)$$

is of class C^n then $u = W$ is of class C^{n+1}.

By the principle of induction, u is therefore of class C^∞.

We have, of course, the following corollary.

1.1.2.2. Corollary

If u is solution to (P_{int}) (or to (P_∞)), then u belongs to $\mathrm{C}^\infty(\mathbb{R}^2)$.

1.1.2.3. Calculation of the partial derivatives of order two of u

This calculation will be used in 4.2.

If u is solution to (P_{int}) we have:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u_0}{\partial x}(x,y) + G(x,y) = \frac{\partial u_0}{\partial x}(x,y) + \int_{f(x)}^{y} F(x,\eta,u(x,\eta))d\eta.$$

Since $F(.,.,u)$ has a partial derivative in x and that f is derivable, we deduce from it that:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,y) &= \frac{\partial^2 u_0}{\partial x^2}(x,y) - f'(x)F(x,f(x),u(x,f(x))) + \\ &\qquad \int_{f(x)}^{y}\left(\frac{\partial F}{\partial x}(x,\eta,u(x,\eta)) + \frac{\partial F}{\partial z}(x,\eta,u(x,\eta))\frac{\partial u}{\partial x}(x,\eta)\right)d\eta \\ &= \frac{\partial^2 u_0}{\partial x^2}(x,y) - f'(x)F(x,f(x),\varphi(x)) + \\ &\qquad \int_{f(x)}^{y}\left(\frac{\partial F}{\partial x}(x,\eta,u(x,\eta)) + \frac{\partial F}{\partial z}(x,\eta,u(x,\eta))\frac{\partial u}{\partial x}(x,\eta)\right)d\eta.\end{aligned}$$

As: $u_0(x,y) = \chi(y) - \chi(f(x)) + \varphi(x)$, it follows succesively

$$\begin{aligned}\frac{\partial u_0}{\partial x}(x,y) &= -f'(x)\psi\left(f^{-1}\left(f(x)\right)\right) + \varphi'(x) = -f'(x)\psi(x) + \varphi'(x) \\ \frac{\partial^2 u_0}{\partial x^2}(x,y) &= -f"(x)\psi(x) - f'(x)\psi'(x) + \varphi"(x).\end{aligned}$$

Then we find:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,y) &= -f"(x)\psi(x) - f'(x)\psi'(x) + \varphi"(x) - f'(x)F(x,f(x),\varphi(x)) \\ &\qquad + \int_{f(x)}^{y}\left(\frac{\partial F}{\partial x}(x,\eta,u(x,\eta)) + \frac{\partial F}{\partial z}(x,\eta,u(x,\eta))\frac{\partial u}{\partial x}(x,\eta)\right)d\eta.\end{aligned}$$

If u is solution to (P_{int})), we can write, supposing that $y \geq f(x)$:

$$u(x,y) = u_0(x,y) - \int_x^{f^{-1}(y)} \left(\int_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))d\eta \right) d\xi.$$

Let us calculate again $u(x,y)$ in the following way:

$$u(x,y) = u_0(x,y) - \int_{f(x)}^{y} \left(\int_x^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))d\xi \right) d\eta.$$

Starting from:

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u_0}{\partial y}(x,y) - \int_x^{f^{-1}(y)} F(\xi,y,u(\xi,y))d\xi$$

we obtain:

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}(x,y) = \frac{\partial^2 u_0}{\partial y^2}(x,y) &- \left(\frac{1}{f'(f^{-1}(y))}\right) F(f^{-1}(y),y,\varphi(f^{-1}(y))) \\ &- \int_x^{f^{-1}(y)} \left(\frac{\partial F}{\partial y}(\xi,y,u(\xi,y)) + \frac{\partial F}{\partial z}(\xi,y,u(\xi,y))\frac{\partial u}{\partial y}(\xi,y) \right) d\xi,\end{aligned}$$

$$u_0(x,y) = \chi\,(y) - \chi(f(x)) + \varphi(x).$$

As:

$$\frac{\partial u_0}{\partial y}(x,y) = \psi(f^{-1}(y)),$$

we have:

$$\frac{\partial^2 u_0}{\partial y^2}(x,y) = \left[(f^{-1})'(y)\right] \psi'\left(f^{-1}(y)\right) = \left[\frac{1}{f'(f^{-1}(y))}\right] \psi'\left(f^{-1}(y)\right)\ \square.$$

1.2 Solving the problem

1.2.1 Solving (P_∞)

1.2.1.1. Theorem

Under hypothesis (H) the problem (P_∞) has a unique solution in $\mathrm{C}^\infty(\mathbb{R}^2)$.

Proof.

According to theorem 1.1.2.1., solving (P_∞) amounts to solving (P_{int}) , that is searching $u \in \mathrm{C}^0(\mathbb{R}^2)$ verifying:

$$u(x,y) = u_0(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta. \ (4)$$

For every compact of $\mathbb{R}^2$, we can find $\lambda > 0$, big enough, so that this compact is contained in $K_\lambda = [-\lambda;\lambda] \times [f(-\lambda); f(\lambda)]$. So the family $(K_\lambda)_{\lambda \in \mathbb{R}_+^*}$ is a exhaustive family of compacts.

a) Change of variables.

Let us assume always that: $y \geq f(x)$ and let us make the change of variables:

$$\begin{cases} X = x + \lambda \\ Y = y - f(-\lambda) \end{cases}$$

that is, let us consider $\Omega(-\lambda, f(-\lambda))$ as new origin .

Then:

$$\begin{cases} x = X - \lambda \\ y = Y + f(-\lambda). \end{cases}$$

The relation (4) is written:

$$u(X-\lambda, Y+f(-\lambda)) = u_0(X-\lambda, Y+f(-\lambda)) - \iint_{D(X-\lambda,Y+f(-\lambda),f)} F(\xi-\lambda, \eta+f(-\lambda), u(\xi-\lambda,\eta+f(-\lambda)))d\xi d\eta,$$

whose form is:

$$U(X,Y) = U_0(X,Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))d\xi d\eta, \qquad (5)$$

with: $g(X) = f(X-\lambda) - f(-\lambda)$; K_λ is turned into the compact $Q_\lambda = [0;2\lambda] \times [0; g(2\lambda)]$ because $g(2\lambda) = f(\lambda) - f(-\lambda)$. The equation of (γ) can then be written: $Y = g(X)$ and $g(0) = 0$.

So we have now:

$$\begin{cases} X \geq 0 \\ \\ Y \geq g(X). \end{cases}$$

b) Solving.

According to the hypothesis (H), we can put:

$$m_\lambda = \sup_{(\xi,\eta)\in Q_\lambda;\ z\in\mathbb{R}} \left| \frac{\partial \mathfrak{F}}{\partial z}(\xi,\eta,z) \right|.$$

Let us define the sequence $(U_n)_{n\in\mathbb{N}}$ of functions of $\mathbb{R}^2$ by:

$$\forall n \in \mathbb{N}^*,\ U_n(X,Y) = U_0(X,Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U_{n-1}(\xi,\eta))d\xi d\eta.$$

For every compact $H \Subset \mathbb{R}^2$, let us put:

$$\|U_0\|_{\infty,H} = \sup_{(x,y)\in H} |U_0(x,y)|.$$

According to the finished increment theorem under its integral form, we can write:

$$\mathfrak{F}(\xi,\eta,t) - \mathfrak{F}(\xi,\eta,r) = (t-r)\int_0^1 \frac{\partial \mathfrak{F}}{\partial z}(\xi,\eta,r+\sigma(t-r))d\sigma, \qquad (6)$$

hence: $\forall(\xi,\eta) \in \mathfrak{D}(X,Y,g)$

$$\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,0) = U_0(\xi,\eta)\int_0^1 \frac{\partial \mathfrak{F}}{\partial z}(\xi,\eta,\sigma U_0(\xi,\eta))d\sigma.$$

So:

$$|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta))| \leq |\mathfrak{F}(\xi,\eta,0)| + |U_0(\xi,\eta)| \int_0^1 m_\lambda d\sigma,$$

$$|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta))| \leq |\mathfrak{F}(\xi,\eta,0)| + m_\lambda \|U_0\|_{\infty,Q_\lambda}.$$

Let us put:

$$\Phi_\lambda = \|\mathfrak{F}(.,.,0)\|_{\infty,Q_\lambda} + m_\lambda \|U_0\|_{\infty,Q_\lambda},$$

$$\forall n \in \mathbb{N}^*,\ V_n = U_n - U_{n-1},$$

which involves:

$$V_1(X,Y) = U_1(X,Y) - U_0(X,Y) = -\iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U_0(\xi,\eta))d\xi d\eta,$$

$$|V_1(X,Y)| \leq \iint_{\mathfrak{D}(X,Y,g)} |\mathfrak{F}(\xi,\eta,U_0(\xi,\eta))|\, d\xi d\eta \leq \Phi_\lambda \iint_{\mathfrak{D}(X,Y,g)} d\xi d\eta,$$

$$|V_1(X,Y)| \leq \Phi_\lambda A(X,Y),$$

where: $A(X,Y) = \int\int_{\mathfrak{D}(X,Y,g)} d\xi d\eta$, indicates the area of the domain $\mathfrak{D}(X,Y,g)$ limited by the "curvilinear triangle MPQ".

We have:

$$\begin{aligned}|V_2(X,Y)| &= |U_2(X,Y) - U_1(X,Y)| \\ &\leq \iint_{\mathfrak{D}(X,Y,g)} |\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_1(\xi,\eta))|\, d\xi d\eta.\end{aligned}$$

Then using the relation (6), we obtain:

$$\begin{aligned}&|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_1(\xi,\eta))| \\ &\qquad \leq |U_0(\xi,\eta) - U_1(\xi,\eta)| \left| \int_0^1 \frac{\partial}{\partial z}\mathfrak{F}(\xi,\eta,U_1(\xi,\eta) + \sigma(U_1(\xi,\eta) - U_0(\xi,\eta)))d\sigma \right|,\end{aligned}$$

consequently:

$$|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_1(\xi,\eta))| \leq |U_0(\xi,\eta) - U_1(\xi,\eta)|\, m_\lambda$$

$$|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_1(\xi,\eta))| \leq |V_1(\xi,\eta)|\, m_\lambda.$$

From this it may be deduced that:

$$\begin{aligned}|V_2(X,Y)| &\leq \iint_{\mathfrak{D}(X,Y,g)} m_\lambda\, |U_0(\xi,\eta) - U_1(\xi,\eta)|\, d\xi d\eta \\ &\leq m_\lambda \iint_{\mathfrak{D}(X,Y,g)} |V_1(\xi,\eta)|\, d\xi d\eta \\ &\leq m_\lambda \Phi_\lambda \iint_{\mathfrak{D}(X,Y,g)} A(\xi,\eta) d\xi d\eta.\end{aligned}$$

We can notice that:

$$A(X,Y) \leq (g^{-1}(Y)-X)(Y-g(X)) \leq (2\lambda - X)Y \leq (2\lambda)Y$$

and then:

$$\begin{aligned}
|V_2(X,Y)| &\leq m_\lambda \Phi_\lambda \iint_{\mathfrak{D}(X,Y,g)} A(\xi,\eta) d\xi d\eta \leq m_\lambda \Phi_\lambda \iint_{\mathfrak{D}(X,Y,g)} (2\lambda)\eta d\xi d\eta \\
&\leq m_\lambda \Phi_\lambda \int_0^Y (\int_0^{2\lambda} (2\lambda)\eta d\xi) d\eta \\
&\leq m_\lambda \Phi_\lambda \left((2\lambda)^2 \frac{Y^2}{2} \right).
\end{aligned}$$

Consequently:

$$\forall (\xi,\eta) \in \mathfrak{D}(X,Y,g),\ |V_2(\xi,\eta)| \leq m_\lambda \Phi_\lambda \left((2\lambda)^2 \frac{\eta^2}{2} \right).$$

In the same way:

$$|V_3(X,Y)| = |U_3(X,Y) - U_2(X,Y)| \leq \iint_{\mathfrak{D}(X,Y,g)} |\mathfrak{F}(\xi,\eta,U_1(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_2(\xi,\eta))| \, d\xi d\eta.$$

Hence:

$$\begin{aligned}
|V_3(X,Y)| &\leq \iint_{\mathfrak{D}(X,Y,g)} m_\lambda \, |U_1(\xi,\eta) - U_2(\xi,\eta)| \, d\xi d\eta \\
&\leq m_\lambda \iint_{\mathfrak{D}(X,Y,g)} |V_2(\xi,\eta)| \, d\xi d\eta \\
&\leq m_\lambda \iint_{\mathfrak{D}(X,Y,g)} m_\lambda \Phi_\lambda \left((2\lambda)^2 \frac{\eta^2}{2} \right) d\xi d\eta \\
&\leq m_\lambda^2 \Phi_\lambda \int_0^Y (\int_0^{2\lambda} (2\lambda)^2 \frac{\eta^2}{2} d\xi) d\eta \\
&\leq m_\lambda^2 \Phi_\lambda \left((2\lambda)^3 \frac{Y^3}{3!} \right).
\end{aligned}$$

It follows by induction:

$$|V_n(X,Y)| \leq m_\lambda^{n-1} \Phi_\lambda \left((2\lambda)^n \frac{Y^n}{n!} \right).$$

Hence:

$$\|V_n\|_{\infty,Q_\lambda} \leq m_\lambda^{n-1} \Phi_\lambda \frac{[(2\lambda) g(2\lambda)]^n}{n!} = \frac{\Phi_\lambda}{m_\lambda} \frac{[(2\lambda) m_\lambda g(2\lambda)]^n}{n!},$$

which ensures the uniform convergence of the series $\sum\limits_{n\geq 1} V_n$ on Q_λ and consequently on every compact of $\mathbb{R}^2$.

The equality $\sum_{k=1}^{n} V_k = U_n - U_0$ involves that the sequence $(U_n)_{n\in\mathbb{N}}$ converges uniformly on Q_λ to a function U. As every U_n is continuous, the uniform limit U is continuous on every compact Q_λ , so on $\mathbb{R}^2$.

Let us put: $\varepsilon_n(X,Y) = U(X,Y) - U_n(X,Y)$, then:

$$\begin{aligned}
&U(X,Y) - U_0(X,Y) + \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))d\xi d\eta \\
&\qquad = U(X,Y) - U_n(X,Y) + \left(U_n(X,Y) - U_0(X,Y) + \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))d\xi d\eta \right) \\
&\qquad = \varepsilon_n(X,Y) + \iint_{\mathfrak{D}(X,Y,g)} \left(\mathfrak{F}(\xi,\eta,U(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_{n-1}(\xi,\eta))\right) d\xi d\eta \ .
\end{aligned}$$

As:

$$\forall(\xi,\eta) \in \mathfrak{D}(X,Y,g) : |\mathfrak{F}(\xi,\eta,U(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_n(\xi,\eta))| \leq |U(\xi,\eta) - U_{n-1}(\xi,\eta)|\, m_\lambda,$$

the second member is bounded by:

$$\left(\sup_{(X,Y)\in Q_\lambda} |\varepsilon_n(X,Y)| \right) + m_\lambda \left(\sup_{(X,Y)\in Q_\lambda} A(X,Y) \right) \left(\sup_{(X,Y)\in Q_\lambda} |U(X,Y) - U_{n-1}(X,Y)| \right),$$

that is by:

$$\left(\sup_{(X,Y)\in Q_\lambda} |\varepsilon_n(X,Y)| \right) + m_\lambda \left(2\lambda \times g(2\lambda)\right) \left(\sup_{(X,Y)\in Q_\lambda} |\varepsilon_{n-1}(X,Y)| \right)$$

whose limit is 0 when n tends to $+\infty$. It follows that:

$$U(X,Y) = U_0(X,Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))d\xi d\eta$$

for $(X,Y) \in Q_\lambda \cap \{(X,Y)/Y \geq g(X)\} = Q_\lambda^+$.

c) Let us show the uniqueness of the solution.

Let W be another solution to:

$$U(X,Y) = U_0(X,Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))d\xi d\eta.$$

Putting: $\Delta = W - U$, we obtain:

$$\Delta(X,Y) = \iint_{\mathfrak{D}(X,Y,g)} \left(-\mathfrak{F}(\xi,\eta,W(\xi,\eta)) + \mathfrak{F}(\xi,\eta,U(\xi,\eta))\right) d\xi d\eta.$$

Let $(X,Y) \in Q_\lambda$, since $\mathfrak{D}(X,Y,g) \subset Q_\lambda$, we have:

$$|\Delta(X,Y)| \leq \iint_{\mathfrak{D}(X,Y,g)} m_\lambda \left|W(\xi,\eta) - U(\xi,\eta)\right| d\xi d\eta \leq m_\lambda \iint_{\mathfrak{D}(X,Y,g)} |\Delta(\xi,\eta)| \, d\xi d\eta.$$

As $Y \geq g(X)$,

$$\begin{aligned} |\Delta(X,Y)| &\leq m_\lambda \int_X^{g^{-1}(Y)} \int_{g(X)}^Y |\Delta(\xi,\eta)| \, d\eta d\xi \leq m_\lambda \int_0^{2\lambda} \int_0^Y |\Delta(\xi,\eta)| \, d\eta d\xi \\ &\leq m_\lambda \int_0^Y (\int_0^{2\lambda} |\Delta(\xi,\eta)| \, d\xi) d\eta \\ &\leq m_\lambda \int_0^Y (\int_0^{2\lambda} \sup_{\xi \in [0;2\lambda]} |\Delta(\xi,\eta)| \, d\xi) d\eta. \end{aligned}$$

For every $Y \in [0; g(2\lambda)]$, let us put:

$$E(Y) = \sup_{\xi \in [0;2\lambda]} |\Delta(\xi,Y)|,$$

then:

$$|\Delta(X,Y)| \leq m_\lambda \left| \int_0^Y \int_0^{2\lambda} E(\eta) d\xi d\eta \right| \leq m_\lambda 2\lambda \left| \int_0^Y E(\eta) d\eta \right|,$$

with the result that:

$$\forall Y \in [0; g(2\lambda)], E(Y) \leq m_\lambda 2\lambda \left| \int_0^Y E(\eta) d\eta \right|,$$

in this way, by applying GRONWALL lemma we get: $E = 0$ hence $\Delta = 0$, which proves the uniqueness of U in Q_λ^+.

Then putting $v_\lambda(x,y) = U(x+\lambda, y - f(-\lambda))$, if follows that v_λ is the unique solution to (4) in $K_\lambda \cap \{(x,y)/y \geq f(x)\} = K_\lambda^+$.

d) Case $y \leq f(x)$

In the case $y \leq f(x)$, let us make the change of variables:

$$\begin{cases} X = -x + \lambda \\ Y = -y + f(\lambda), \end{cases}$$

that is, let be consider $\Omega(\lambda, f(\lambda))$ as new origin and let us reverse the direction of the axis.

Then:

$$\begin{cases} x = -X + \lambda \\ y = -Y + f(\lambda) \end{cases}$$

and:

$$u(-X+\lambda, -Y+f(\lambda)) = u_0(-X+\lambda, -Y+f(\lambda)) - \iint_{D(-X+\lambda,-Y+f(\lambda),f)} F(-\xi+\lambda, -\eta+f(\lambda), u(-\xi+\lambda, -\eta+f(\lambda)))d\xi d\eta,$$

whose form is:

$$W(X,Y) = W_0(X,Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi, \eta, W(\xi,\eta))d\xi d\eta$$

and: $g(X) = f(\lambda) - f(\lambda - X)$; K_λ is turned into the compact $Q_\lambda = [0; 2\lambda] \times [0; g(2\lambda)]$ because $g(2\lambda) = f(\lambda) - f(-\lambda)$.

As $y \leq f(x)$ is equivalent to: $-y \geq -f(x)$, so we have:

$$f(\lambda) - y \geq f(\lambda) - f(x),$$

then: $Y \geq f(\lambda) - f(\lambda - X)$, that is to say: $Y \geq g(X)$. We boil down to the case:

$$\begin{cases} X \geq 0 \\ Y \geq g(X) \end{cases}$$

which can be dealt with like the previous case.

e) Solution.

The result of this is that $w_\lambda(x,y) = W(-x+\lambda, -y+f(\lambda))$ is solution to (4) in:

$$K_\lambda \cap \{(x,y)/y \leq f(x)\} = K_\lambda^-.$$

From the continuity of U in Q_λ^+ and of W in Q_λ^- we have as result the continuity of v_λ in K_λ^+ and of w_λ in K_λ^-. Moreover, v_λ and w_λ link up on γ because $v_\lambda(x,f(x)) = w_\lambda(x,f(x)) = \varphi(x)$.

Finally if we put:

$$u_\lambda(x,y) = \begin{cases} v_\lambda(x,y) \text{ for } (x,y) \in K_\lambda^+ \\ \\ w_\lambda(x,y) \text{ for } (x,y) \in K_\lambda^- \end{cases}$$

then u_λ is the unique solution to (P_{int}) continuous on K_λ.

It remains to prove that the method actually gives a continuous global solution u to (4) in $\mathbb{R}^2$, that is, which verifies (P_{int}).

If $\lambda_2 > \lambda_1$ then $K_{\lambda_1} \subset K_{\lambda_2}$: so, we must still prove that $u_{\lambda_2|K_{\lambda_1}} = u_{\lambda_1}$.

But:

$$\forall (x,y) \in K_{\lambda_2},\, u_{\lambda_2}(x,y) = u_0(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u_{\lambda_2}(\xi,\eta))d\xi d\eta$$

and we have this equality all the more for $(x,y) \in K_{\lambda_1}$. So we have:

$$u_{\lambda_{2|K_{\lambda_1}}}(x,y) = u_0(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u_{\lambda_{2|K_{\lambda_1}}}(\xi,\eta))d\xi d\eta.$$

In other words: $u_{\lambda_{2|K_{\lambda_1}}}$ verifies (4) in K_{λ_1} and so coincides on it with its unique solution u_{λ_1}.

For every $(x,y) \in \mathbb{R}^2$ then we can put:

$$\begin{aligned} u(x,y) &= u_\lambda(x,y) \\ &= u_0(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta, \quad (7) \end{aligned}$$

where u_λ verifies (4) in K_λ and $(x,y) \in K_\lambda$.

The definition of u in (7) being independent of the compact K_λ finally gives the unique global solution to (P_{int}) or (P_∞). $\square$

In chapter 4, we will need the estimations clarified by the following result.

1.2.1.2. Proposition.

With the previous notations, for every compact $K \Subset R^2$*, there exits a compact* K_λ, $K_\lambda \Subset R^2$*, defined before, containing* K*, such that:*

(i) $m_\lambda = \sup\limits_{(x,y)\in K_\lambda;\ t\in\mathbb{R}} \left|\frac{\partial F}{\partial z}(x,y,t)\right|$; $\Phi_\lambda = \|F(.,.,0)\|_{\infty,K_\lambda} + m_\lambda \|u_0\|_{\infty,K_\lambda}$;

(ii) $\|u\|_{\infty,K} \leq \|u\|_{\infty,K_\lambda} \leq \|u_0\|_{\infty,K_\lambda} + \dfrac{\Phi_\lambda}{m_\lambda}\exp[2\lambda m_\lambda (f(\lambda) - f(-\lambda))]$.

Proof.

(i) We have clearly:

$$m_\lambda = \sup_{(\xi,\eta)\in Q_\lambda;\ t\in\mathbb{R}} \left|\frac{\partial \mathfrak{F}}{\partial z}(\xi,\eta,t)\right| = \sup_{(x,y)\in K_\lambda;\ t\in\mathbb{R}} \left|\frac{\partial F}{\partial z}(x,y,t)\right|;$$

$$\Phi_\lambda = \|\mathfrak{F}(.,.,0)\|_{\infty,Q_\lambda} + m_\lambda \|U_0\|_{\infty,Q_\lambda} = \|F(.,.,0)\|_{\infty,K_\lambda} + m_\lambda \|u_0\|_{\infty,K_\lambda}.$$

(ii) Keeping the previous notations, we have:

$$u_n(x,y) = u_0(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u_{n-1}(\xi,\eta))d\xi d\eta, \ \ n \geq 1,$$

$$u_0(x,y) = \chi(y) - \chi(f(x)) + \varphi(x).$$

$$u_{n,\lambda}(x,y) = \begin{cases} v_{n,\lambda}(x,y) \text{ for } (x,y) \in K_\lambda^+ \\ w_{n,\lambda}(x,y) \text{ for } (x,y) \in K_\lambda^-. \end{cases}$$

As:

$$U_n(X,Y) = U_0(X,Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U_{n-1}(\xi,\eta))d\xi d\eta,$$

$$\Phi_\lambda = \|\mathfrak{F}(.,.,0)\|_{\infty,Q_\lambda} + m_\lambda \|U_0\|_{\infty,Q_\lambda},$$

$$V_n = U_n - U_{n-1},$$

where K_λ is turned by g into the compact $Q_\lambda = [0; 2\lambda] \times [0; g(2\lambda)]$, with: $K_\lambda = [-\lambda; \lambda] \times [f(-\lambda); f(\lambda)]$.

$$\begin{cases} \text{if } y \geq f(x),\ g(X) = f(X-\lambda) - f(-\lambda), \\ \text{if } y \leq f(x),\ g(X) = f(\lambda) - f(\lambda - X). \end{cases}$$

According to the proof of theorem 1.2.1.1, we have:

$$\forall n \in \mathbb{N}^*, \quad \|V_n\|_{\infty, Q_\lambda} \leq m_\lambda^{n-1} \Phi_\lambda \frac{[(2\lambda) g(2\lambda)]^n}{n!} = \frac{\Phi_\lambda}{m_\lambda} \frac{[m_\lambda (2\lambda) g(2\lambda)]^n}{n!},$$

and consequently:

$$\|U\|_{\infty, Q_\lambda} \leq \|U_0\|_{\infty, Q_\lambda} + \sum_{n=1}^{\infty} \|V_n\|_{\infty, Q_\lambda} \leq \|U_0\|_{\infty, Q_\lambda} + \frac{\Phi_\lambda}{m_\lambda} \exp[m_\lambda (2\lambda) g(2\lambda)].$$

Furthermore:

$$g(2\lambda) = f(\lambda) - f(-\lambda).$$

As we have the following relations:

$$\begin{cases} \|v_\lambda\|_{\infty, K_\lambda^+} = \|U\|_{\infty, Q_\lambda} \\ \|w_\lambda\|_{\infty, K_\lambda^-} = \|W\|_{\infty, Q_\lambda} \end{cases}, \quad \begin{cases} \|u_0\|_{\infty, K_\lambda^+} = \|U_0\|_{\infty, Q_\lambda} \\ \|u_0\|_{\infty, K_\lambda^-} = \|W_0\|_{\infty, Q_\lambda} \end{cases}, \; u_\lambda = \begin{cases} v_\lambda \text{ on } K_\lambda^+ \\ w_\lambda \text{ on } K_\lambda^- \end{cases},$$

from this it may be deduced that:

$$\|u\|_{\infty, K_\lambda^+} \leq \|u_0\|_{\infty, K_\lambda^+} + \frac{\Phi_\lambda}{m_\lambda} \exp[m_\lambda (2\lambda)\,(f(\lambda) - f(-\lambda))],$$

and, in the same way:

$$\|u\|_{\infty, K_\lambda^-} \leq \|u_0\|_{\infty, K_\lambda^-} + \frac{\Phi_\lambda}{m_\lambda} \exp[m_\lambda (2\lambda)\,(f(\lambda) - f(-\lambda))].$$

So:

$$\|u\|_{\infty, K_\lambda} \leq \|u_0\|_{\infty, K_\lambda} + \frac{\Phi_\lambda}{m_\lambda} \exp[m_\lambda (2\lambda)\,(f(\lambda) - f(-\lambda))].$$

As $\|u\|_{\infty, K} \leq \|u\|_{\infty, K_\lambda}$, the previous inequality involves the conclusion (ii). □

Chapter 2

A Goursat problem with regular data

2.1 Terms of the problem

2.1.1 Two formulations (P'_∞) and (P'_{int})

F is a function smooth of its arguments and g, φ, ψ are some one-variable smooth real-valued functions with moreover:

$$\psi(0) = \varphi(g(0)).$$

In all cases the following hypothesis will be satisfied:

$$(H) : \begin{cases} F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R}) \\ \forall K \Subset \mathbb{R}^2, \quad \sup\limits_{(x,y)\in K; z\in\mathbb{R}} |\partial_z F(x,y,z)| < +\infty \\ g \text{ is increasing on } \mathbb{R}, \end{cases}$$

where the notation $K \Subset \mathbb{R}^2$ means that K is a compact of $\mathbb{R}^2$.

We denote by (P'_∞) the problem which consits in searching a function $u \in \mathrm{C}^2(\mathbb{R})$ verifying:

$$(P'_\infty)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y)) & (1) \\ u\left(x,0\right)) = \varphi(x) & (2) \\ u(g\left(y\right),y) = \psi(y). & (3) \end{cases}$$

We denote by (P'_{int}) the problem which consits in searching a function $u \in \mathrm{C}^0(\mathbb{R})$ verifying

$$u(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta, \qquad (4)$$

where:

$$u_0(x,y) = \psi(y) + \varphi(x) - \varphi(g(y)),$$

with:

$$D(x,y,g) = \begin{cases} \{(\xi,\eta),\ g(y) \leq \xi \leq x;\ 0 \leq \eta \leq y\} \text{ if } g(y) \leq x \text{ and } 0 \leq y \\ \{(\xi,\eta), x \leq \xi \leq g(y);\ 0 \leq \eta \leq y\} \text{ if } g(y) \geq x \text{ and } 0 \leq y \\ \{(\xi,\eta), x \leq \xi \leq g(y);\ y \leq \eta \leq 0\} \text{ if } g(y) \geq x \text{ and } y \leq 0 \\ \{(\xi,\eta),\ g(y) \leq \xi \leq x;\ y \leq \eta \leq 0\} \text{ if } g(y) \leq x \text{ and } y \leq 0. \end{cases}$$

2.1.2 Equivalence of the two formulations

2.1.2.1. Theorem

Let $u \in \mathrm{C}^0(\mathbb{R}^2)$. *The function* u *is solution to* (P'_∞) if and only if u is *solution to* (P'_{int}).

Proof.

Necessary condition.

We consider the points $M(x,y)$, $N(x,0)$, $P(g(y),y)$, $Q(g(y),0)$.

The hypothesis (H) assure that domain $D(x,y,g)$ is bounded.

a) Let us suppose first: $0 \leq y$ and $g(y) \leq x$.

$D(x,y,g)$ is limited by rectangle $PQNM$.

$$\begin{aligned}\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\xi d\eta &= \int_{g(y)}^{x} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi = \int_{g(y)}^{x} \left[\frac{\partial u}{\partial x}(\xi,\eta) \right]_0^y d\xi \\ &= \int_{g(y)}^{x} \frac{\partial u}{\partial x}(\xi,y)d\xi - \int_{g(y)}^{x} \frac{\partial u}{\partial x}(\xi,0))d\xi \\ &= [u(\xi,y)]_{g(y)}^{x} - [\varphi(\xi)]_{g(y)}^{x} \\ &= u(x,y) - u(g(y),y) - \varphi(x) + \varphi(g(y)) \\ &= u(x,y) - \psi(y) - \varphi(x) + \varphi(g(y)).\end{aligned}$$

We deduce that:

$$\begin{aligned}u(x,y) &= \psi(y) + \varphi(x) - \varphi(g(y)) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta \\ &= u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta,\end{aligned}$$

where:

$$u_0(x,y) = \psi(y) + \varphi(x) - \varphi(g(y)).$$

So we have: $u_0(x,0) = \psi(0) + \varphi(x) - \varphi(g(0))$, and, as: $u(x,0) = \varphi(x)$, it results that:

$$u_0(g(y),y) = \psi(y) + \varphi(g(y)) - \varphi(g\,(y)) = \psi(y).$$

So u is solution to (P'_{int}).

To calculate:

$$\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\xi d\eta$$

we must distinguish four cases:

the case (1) : $(0 \leq y$ and $g\ \ (y) \leq x)$,

the case (2): $(0 \leq y$ and $x \leq g(y))$,

the case (3): $(\ y \leq 0$ and $x \leq g\,(y)\,$,

the case (4): $(y \leq 0$ and $g(y) \leq x)$.

Let us briefly consider the other cases:

b) Case (2): If ($0 \leq y$ and $x \leq g\left(y\right)$), then:

$$\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\xi d\eta = \int_{x}^{g(y)} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi = -\int_{g(y)}^{x} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi.$$

Case (3): If ($x \leq g(y)$ and $y \leq 0$), then:

$$\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\xi d\eta = \int_{x}^{g(y)} \left(\int_y^0 \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi = \int_{g(y)}^{x} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi.$$

Case (4): If ($y \leq 0$ and $g(y) \leq x$), then:

$$\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\xi d\eta = \int_{g(y)}^{x} \left(\int_y^0 \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi = -\int_{g(y)}^{x} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi.$$

Sufficient condition.

If u verifies (P'_{int}), assume that $g\left(y\right) \leq x$ and $0 \leq y$, we can write:

$$u(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta.$$

$$\begin{aligned} u(x,y) &= \int_{g(y)}^{x} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)d\eta \right) d\xi = u_0(x,y) + \int_{g(y)}^{x} \left(\int_0^y F(\xi,\eta,u(\xi,\eta))d\eta \right) d\xi \\ &= u_0(x,y) + \int_{g(y)}^{x} G(\xi,y)d\xi, \end{aligned}$$

where $G(\xi,y) = \int_0^y F(\xi,\eta,u(\xi,\eta))d\eta$. G is a continuous function of ξ and y, so:

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u_0}{\partial x}(x,y) + G(x,y) = \frac{\partial u_0}{\partial x}(x,y) + \int_0^y F(x,\eta,u(x,\eta))d\eta$$

and consequently:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y) = \frac{\partial^2 u_0}{\partial y \partial x}(x,y) + F(x,y,u(x,y))$$

and as:

$$\frac{\partial^2 u_0}{\partial y \partial x}(x,y) = 0,$$

we have:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y)=F(x,y,u(x,y)).$$

Let us calculate again $u(x,y)$ in the following way:

$$u(x,y)=u_0(x,y)+\int_0^y\left(\int_{g(y)}^x F(\xi,\eta,u(\xi,\eta))d\xi\right)d\eta.$$

We have:

$$\frac{\partial u}{\partial y}(x,y)=\frac{\partial u_0}{\partial y}(x,y)+\int_{g(y)}^x F(\xi,y,u(\xi,y))d\xi-g'(y)\int_0^y F(g(y),\eta,u(g(y),\eta))d\eta,$$

hence:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)(x,y)=\frac{\partial^2 u_0}{\partial x\partial y}(x,y)+F(x,y,u(x,y))=F(x,y,u(x,y)).$$

Finally, we can reverse the order of the partial derivatives and:

$$\frac{\partial^2 u}{\partial x\partial y}(x,y)=F(x,y,u(x,y)).$$

Furthermore:

$$u(g(y),y)=u_0(g(y),y)=\psi(y),$$

$$u\left(x,0\right)=u_0\left(x,0\right)=\varphi(x).$$

These results are unchanged if we suppose: $x\leq g\left(y\right)$ and $0\leq y$, so u actually verify (P'_∞).

If u is of class C^1, the function $(x,y)\mapsto F(x,y,u(x,y))$ is of class C^1, then:

$$W:(x,y)\mapsto u_0(x,y)+\int_{g(y)}^x\left(\int_0^y F(\xi,\eta,u(\xi,\eta))d\eta\right)d\xi=u_0(x,y)+\int_{g(y)}^x G(\xi,y)d\xi,$$

have a partial derivative in x of class C^1, and:

$$W:(x,y)\mapsto u_0(x,y)+\int_0^y\left(\int_{g(y)}^x F(\xi,\eta,u(\xi,\eta))d\xi\right)d\eta=u_0(x,y)+\int_0^y H\left(x,\eta\right)d\eta$$

have a partial derivative in y of class C^1.

As:

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x,y)=F(x,y,u(x,y))=\frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x,y),$$

then $u = W$ is of class C^2.

We remark moreover that:

if u is of class C^n, the function $(x, y) \mapsto F(x, y, u(x, y))$ is of class C^n, then:

$$W : (x, y) \mapsto u_0(x, y) + \int_{g(y)}^{x} \left(\int_0^y F(\xi, \eta, u(\xi, \eta)) d\eta \right) d\xi = u_0(x, y) + \int_{g(y)}^{x} G(\xi, y) d\xi,$$

have a partial derivative in x of class C^n, and:

$$W : (x, y) \mapsto u_0(x, y) + \int_0^y \left(\int_{g(y)}^{x} F(\xi, \eta, u(\xi, \eta)) d\xi \right) d\eta = u_0(x, y) + \int_0^y H(x, \eta) \, d\eta$$

have a partial derivative in y of class C^n.

As:

$$\frac{\partial}{\partial x} \left(\frac{\partial W}{\partial y} \right) (x, y) = F(x, y, u(x, y)) = \frac{\partial}{\partial y} \left(\frac{\partial W}{\partial x} \right) (x, y)$$

is of class C^n then $u = W$ is of class C^{n+1}.

By the principle of induction, u is therefore of class C^∞. □

We have, of course, the following corollary.

2.1.2.2. Corollary

If u is solution to (P'_{int}) (or to (P'_∞)), then u belongs to $\mathrm{C}^\infty(\mathbb{R}^2)$.

2.1.2.3. Calculation of the partial derivatives of order two of u

This calculation will be used in 6.2.

Let us assume that u is solution to (P'_{int}), $g(y) \leq x$ and $0 \leq y$.

Let us remember that:

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial u_0}{\partial x}(x, y) + G(x, y) = \frac{\partial u_0}{\partial x}(x, y) + \int_0^y F(x, \eta, u(x, \eta)) d\eta.$$

As: $\dfrac{\partial^2 u_0}{\partial x^2}(x, y) = \varphi''(x)$, we find then:

$$\frac{\partial^2 u}{\partial x^2}(x, y) = \varphi''(x) + \int_0^y \left(\frac{\partial F}{\partial x}(x, \eta, u(x, \eta)) + \frac{\partial F}{\partial z}(x, \eta, u(x, \eta)) \frac{\partial u}{\partial x}(x, \eta) \right) d\eta.$$

We calculate again $u(x, y)$ in the following way:

$$u(x,y) = u_0(x,y) + \int_0^y \left(\int_{g(y)}^x F(\xi, \eta, u(\xi, \eta)) d\xi \right) d\eta.$$

Starting from:

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u_0}{\partial y}(x,y) + \int_{g(y)}^x F(\xi, y, u(\xi, y)) d\xi - g'(y) \int_0^y F(g(y), \eta, u(g(y), \eta)) d\eta$$

we obtain then:

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2}(x,y) &= \frac{\partial^2 u_0}{\partial y^2}(x,y) - 2g'(y) F(g(y), y, u(g(y), y)) \\
&\qquad - \int_x^{g(y)} \left(\frac{\partial F}{\partial y}(\xi, y, u(\xi, y)) + \frac{\partial F}{\partial z}(\xi, y, u(\xi, y)) \frac{\partial u}{\partial y}(\xi, y) \right) d\xi, \\
u_0(x,y) &= \psi(y) + \varphi(x) - \varphi(g(y)), \\
\frac{\partial u_0}{\partial y}(x,y) &= \psi'(y) - \left[g'(y) \right] \varphi'(g(y)).
\end{aligned}$$

Hence:

$$\frac{\partial^2 u_0}{\partial y^2}(x,y) = \psi''(y) - \left(g''(y) \varphi'(g(y)) + \left(g'(y) \right)^2 \varphi''(g(y)) \right).$$

□

2.2 Solving the problem

2.2.1 Solving (P'_∞)

2.2.1.1. Theorem

Under hypothesis (H) problem (P'_∞) has a unique solution u in $\mathrm{C}^\infty(\mathbb{R}^2)$.

Proof.

a) Let us assume that: $0 \leq y$.

a1) Case (1) $g(y) \leq x$.

According to the theorem 2.1.2.1., solving (P'_{∞}) amounts solving (P'_{int}), that is searching $u \in \mathrm{C}^0(\mathbb{R})$ verifying: $u(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi, \eta, u(\xi,\eta)) d\xi d\eta, \qquad (4)$

a11) Existence.

For every compact of $\mathbb{R}^2$, we can find λ, big enough, so that this compact is contained in $K_\lambda = [g(-\lambda); g(\lambda)] \times [-\lambda; \lambda]$

We search u such that:

$$u(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta), u(\xi,\eta)) d\xi d\eta.$$

Let us put, in accordance with the hypothesis (H):

$$m_\lambda = \sup_{(\xi,\eta)\in K_\lambda;\ z\in\mathbb{R}} \left| \frac{\partial F}{\partial z}(\xi,\eta,z) \right|.$$

Let us define the sequence $(u_n)_{n\in\mathbb{N}}$ of functions of $\mathbb{R}^2$ by:

$$\forall n \in \mathbb{N}^*,\ u_n(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u_{n-1}(\xi,\eta)) d\xi d\eta.$$

Let us put for every compact $H \Subset \mathbb{R}^2$:

$$\|u_0\|_{\infty,H} = \sup_{(x,y)\in H} |u_0(x,y)|.$$

According to the finished increment theorem under its integral form, we can write:

$$F(\xi,\eta,t) - F(\xi,\eta,r) = (t-r)\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,r+\sigma(t-r))d\sigma, \qquad (5)$$

hence, for every $(\xi,\eta) \in D(x,y,g)$, we have:

$$F(\xi,\eta,u_0(\xi,\eta)) - F(\xi,\eta,0) = u_0(\xi,\eta)\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,\sigma u_0(\xi,\eta))d\sigma,$$

$$|F(\xi,\eta,u_0(\xi,\eta))| \leq |F(\xi,\eta,0)| + |u_0(\xi,\eta)| \int_0^1 m_\lambda d\sigma,$$

$$|F(\xi,\eta,u_0(\xi,\eta))| \leq |F(\xi,\eta,0)| + m_\lambda \|u_0\|_{\infty,K_\lambda}.$$

Let us put:

$$\Phi_\lambda = \|F(.,.,0)\|_{\infty,K_\lambda} + m_\lambda \|u_0\|_{\infty,K_\lambda}\,,$$

$$\forall n \in \mathbb{N}^*,\; V_n = u_n - u_{n-1}.$$

With the notations we have:

$$V_1(x,y) = u_1(x,y) - u_0(x,y) = \iint_{D(x,y,g)} F(\xi,\eta,u_0(\xi,\eta))d\xi d\eta,$$

$$|V_1(x,y)| \leq \iint_{D(x,y,g)} |F(\xi,\eta,u_0(\xi,\eta))|\, d\xi d\eta \leq \Phi_\lambda \iint_{D(x,y,g)} d\xi d\eta,$$

$$|V_1(x,y)| \leq \Phi_\lambda A(x,y),$$

where: $A(x,y) = \int\int_{D(x,y,g)} d\xi d\eta$ indicates the area of the domain $D(x,y,g)$.

Just like this, we have:

$$|V_2(x,y)| = |u_2(x,y) - u_1(x,y)| \leq \iint_{D(x,y,g)} |F(\xi,\eta,u_1(\xi,\eta)) - F(\xi,\eta,u_0(\xi,\eta))|\, d\xi d\eta.$$

Then using the relation (5), we obtain:

$$\begin{aligned}
&|F(\xi,\eta,u_1(\xi,\eta)) - F(\xi,\eta,u_0(\xi,\eta))| \\
&\qquad \leq |u_1(\xi,\eta) - u_0(\xi,\eta)| \times \left| \int_0^1 \frac{\partial}{\partial z} F(\xi,\eta,u_1(\xi,\eta) + \sigma(u_1(\xi,\eta) - u_0(\xi,\eta)))d\sigma \right| \\
&\qquad \leq |u_1(\xi,\eta) - u_0(\xi,\eta)|\, m_\lambda \\
&\qquad \leq |V_1(\xi,\eta)|\, m_\lambda\,.
\end{aligned}$$

We deduct:

$$\begin{aligned}
|V_2(x,y)| &\leq \iint_{D(x,y,g)} m_\lambda\, |u_1(\xi,\eta) - u_0(\xi,\eta)|\, d\xi d\eta \\
&\leq m_\lambda \iint_{D(x,y,g)} |V_1(\xi,\eta)|\, d\xi d\eta \leq m_\lambda \Phi_\lambda \iint_{D(x,y,g)} A(\xi,\eta) d\xi d\eta.
\end{aligned}$$

We can notice that:

$$A(x,y) \leq |g(y) - x|\, y \leq (g(\lambda) - g(-\lambda))\, y,$$

and, with $(g(\lambda) - g(-\lambda)) = 2\lambda'$, we obtain:

$$\begin{aligned}
|V_2(x,y)| &\leq m_\lambda \Phi_\lambda \iint_{D(x,y,g)} A(\xi,\eta) d\xi d\eta \leq m_\lambda \Phi_\lambda \iint_{D(x,y,g)} (2\lambda')\eta d\xi d\eta \\
&\leq m_\lambda \Phi_\lambda \int_0^y (\int_0^{2\lambda} (2\lambda')\eta d\xi) d\eta \\
&\leq m_\lambda \Phi_\lambda \left((2\lambda')^2 \frac{y^2}{2} \right).
\end{aligned}$$

Consequently:

$$\forall (\xi,\eta) \in D(x,y,g),\ |V_2(\xi,\eta)| \leq m_\lambda \Phi_\lambda \left((2\lambda')^2 \frac{\eta^2}{2} \right).$$

Just like this, we also have:

$$|V_3(x,y)| = |u_3(x,y) - u_2(x,y)| \leq \iint_{D(x,y,g)} |F(\xi,\eta,u_1(\xi,\eta)) - F(\xi,\eta,u_2(\xi,\eta))| \, d\xi d\eta,$$

$$\begin{aligned}
|V_3(x,y)| &\leq \iint_{D(x,y,g)} m_\lambda \left| u_1(\xi,\eta) - u_2(\xi,\eta) \right| d\xi d\eta \leq m_\lambda \iint_{D(x,y,g)} |V_2(\xi,\eta)| \, d\xi d\eta \\
&\leq m_\lambda \iint_{D(x,y,g)} m_\lambda \Phi_\lambda \left((2\lambda')^2 \frac{\eta^2}{2} \right) d\xi d\eta \\
&\leq m_\lambda^2 \Phi_\lambda \int_0^y (\int_{g-(\lambda)}^{g(\lambda)} (2\lambda')^2 \frac{\eta^2}{2} d\xi) d\eta \leq m_\lambda^2 \Phi_\lambda \left((2\lambda')^3 \frac{y^3}{3!} \right).
\end{aligned}$$

It follows by induction:

$$|V_n(x,y)| \leq m_\lambda^{n-1} \Phi_\lambda \left((2\lambda')^n \frac{y^n}{n!} \right).$$

Hence:

$$\|V_n\|_{\infty,K_\lambda} \leq m_\lambda^{n-1} \Phi_\lambda \frac{[(2\lambda')\lambda]^n}{n!} = \frac{\Phi_\lambda}{m_\lambda} \frac{[(2\lambda') m_\lambda \lambda]^n}{n!}$$

which ensures the uniform convergence of the series $\sum\limits_{n\geq 1} V_n$ on K_λ , and, consequently, on every compact of $\mathbb{R}^2$.

We have: $\sum_{k=1}^{k=n} V_k = u_n - u_0$, so the sequence $(u_n)_{n\in\mathbb{N}}$ converges uniformly on K_λ to a function u. As every u_n is continuous, the uniform limit u is continuous on every compact K_λ, so on $\mathbb{R}^2$.

Let us put $\varepsilon_n(x,y) = u(x,y) - u_n(x,y)$ then:

$$\begin{aligned} u(x,y) - u_0(x,y) - \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta \\ &= u(x,y) - u_n(x,y) + \left(u_n(x,y) - u_0(x,y) - \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta \right) \\ &= \varepsilon_n(x,y) - \iint_{D(x,y,g)} \left(F(\xi,\eta,u(\xi,\eta)) - F(\xi,\eta,u_{n-1}(\xi,\eta)) \right) d\xi d\eta \ , \end{aligned}$$

as for every $(\xi,\eta) \in D(x,y,g)$:

$$|F(\xi,\eta,u(\xi,\eta)) - F(\xi,\eta,u_n(\xi,\eta))| \le |u(\xi,\eta) - u_{n-1}(\xi,\eta)|\, m_\lambda,$$

the second member of the above equality is bounded by:

$$\left(\sup_{(x,y)\in K_\lambda} |\varepsilon_n(x,y)| \right) + m_\lambda \left(\sup_{(x,y)\in K_\lambda} A(x,y) \right) \left(\sup_{(x,y)\in K_\lambda} |u(x,y) - u_{n-1}(x,y)| \right),$$

that is by:

$$\left(\sup_{(x,y)\in K_\lambda} |\varepsilon_n(x,y)| \right) + m_\lambda \left(2\lambda' \times \lambda \right) \left(\sup_{(x,y)\in K_\lambda} |\varepsilon_{n-1}(x,y)| \right),$$

whose limit is 0 when n tends to $+\infty$. So, it follows that:

$$u(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta$$

for $(x,y) \in K_\lambda \cap \{(x,y), 0 \le y, g(y) \le x\} = K^-_{1,\lambda}$.

a12) Uniqueness.

Let W be another solution to:

$$u(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta.$$

Putting: $\Delta = W - u$, we obtain:

$$\Delta(x,y) = \iint_{D(x,y,g)} \left(F(\xi,\eta,W(\xi,\eta)) - F(\xi,\eta,u(\xi,\eta)) \right) d\xi d\eta.$$

Let $(x, y) \in K_\lambda$, since $D(x, y, g) \subset K_\lambda$, we have:

$$|\Delta(x,y)| \leq \iint_{D(x,y,g)} m_\lambda \left|W(\xi,\eta) - u(\xi,\eta)\right| d\xi d\eta \leq m_\lambda \iint_{D(x,y,g)} |\Delta(\xi,\eta)| \, d\xi d\eta.$$

As $g(y) \leq x$

$$\begin{aligned} |\Delta(x,y)| &\leq m_\lambda \int_{g(y)}^{x} \int_0^y |\Delta(\xi,\eta)| \, d\eta d\xi \leq m_\lambda \int_{g(-\lambda)}^{g(\lambda)} \int_0^y |\Delta(\xi,\eta)| \, d\eta d\xi \\ &\leq m_\lambda \int_0^y (\int_{g(-\lambda)}^{g(\lambda)} |\Delta(\xi,\eta)| \, d\xi) d\eta \\ &\leq m_\lambda \int_0^y (\int_{g(-\lambda)}^{g(\lambda)} \sup_{\xi \in [0;2\lambda]} |\Delta(\xi,\eta)| \, d\xi) d\eta. \end{aligned}$$

For every $y \in [0, \lambda]$, let us put:

$$E(y) = \sup_{\xi \in [0;2\lambda]} |\Delta(\xi, y)|,$$

then:

$$|\Delta(x,y)| \leq m_\lambda \left| \int_0^y \int_{-g(-\lambda)}^{g(\lambda)} E(\eta) d\xi d\eta \right| \leq m_\lambda 2\lambda' \left| \int_0^y E(\eta) d\eta \right|,$$

with the result that:

$$\forall Y \in y \in [0; f(\lambda)], E(y) \leq m_\lambda 2\lambda' \left| \int_0^y E(\eta) d\eta \right|,$$

in this way, by applying GRONWALL lemma, we get: $E = 0$ hence $\Delta = 0$, which proves the uniqueness of u in $K_{1,\lambda}^-$. We write v_λ^- this solution.

a2) Case (2): $x \leq g(y)$.

We have:

$$\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta) d\xi d\eta = \int_x^{g(y)} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta) d\eta \right) d\xi = -\int_{g(y)}^{x} \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta) d\eta \right) d\xi.$$

We can treat this case in a same way.

b) $y \leq 0$.

In the case $y \leq 0$, let us make the change of variables:

$$\begin{cases} X = -x \\ \\ Y = -y, \end{cases}$$

that is, we reverse the direction of the axis (symmetry of center O). Then we have: $Y \geq 0$ and $h(Y) = -g(-Y)$. The compact K_λ is turned into the compact $Q_\lambda = [h(-\lambda); h(\lambda)] \times [-\lambda; \lambda]$ and $h(\lambda) = -g(-\lambda)$.

So we have now:

$$\begin{cases} g\,(y) \leq x \Leftrightarrow Y \geq h(X);\ \mathfrak{D}(X, Y, h) = D(-X, -Y, g) = rectangle(MNQP); \\ \\ g\,(y) \geq x \Leftrightarrow Y \leq h(X);\ \mathfrak{D}(X, Y, h) = D(-X, -Y, g) = rectangle(MPQN). \end{cases}$$

b1) If $x \leq g\,(y)$, then:

$$\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi, \eta) d\xi d\eta = \int_x^{g(y)} \left(\int_y^0 \frac{\partial^2 u}{\partial x \partial y}(\xi, \eta) d\eta \right) d\xi = \int_{g(y)}^x \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi, \eta) d\eta \right) d\xi.$$

b2) If $g\,(y) \leq x$, then:

$$\iint_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi, \eta) d\xi d\eta = \int_{g(y)}^x \left(\int_y^0 \frac{\partial^2 u}{\partial x \partial y}(\xi, \eta) d\eta \right) d\xi = -\int_{g(y)}^x \left(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi, \eta) d\eta \right) d\xi.$$

The change of variables give then:

$$u(x, y) = u(-X, -Y) = u_0(-X, -Y) + \iint_{D(-X,-Y,g)} F(-\xi, -\eta, u(-\xi, -\eta)) d\xi d\eta,$$

whose form is:

$$U(X, Y) = U_0(X, Y) + \iint_{\mathfrak{D}(X,Y,h)} \mathfrak{F}(\xi, \eta, u(\xi, \eta)) d\xi d\eta.$$

We can treat the solving like the above cases, with:

$$\|V_n\|_{\infty,K_\lambda} \leq m_\lambda^{n-1}\Phi_\lambda \frac{[(2\lambda)h(\lambda)]^n}{n!} = \frac{\Phi_\lambda}{m_\lambda}\frac{[(2\lambda')m_\lambda\lambda]^n}{n!} \leq \frac{\Phi_\lambda}{m_\lambda}\frac{[(2\lambda')m_\lambda\lambda]^n}{n!},$$

$$u(x,y) = U(-x,-y).$$

c) Existence of a global solution.

We have four cases:

the case : $(0 \leq y$ and $g(y) \leq x)$ and the case: $(0 \leq y$ and $x \leq g(y))$,

the case : $(y \leq 0$ and $x \leq g(y))$ and the case: $(y \leq 0$ and $g(y) \leq x)$.

Finally, if we put:

$$K_{1,\lambda}^{-} = K_\lambda \cap \{(x,y), 0 \leq y \text{ and } g(y) \leq x\},$$

$$K_{2,\lambda}^{+} = K_\lambda \cap \{(x,y), y \leq 0 \text{ and } x \leq g(y)\},$$

$$K_{1,\lambda}^{+} = K_\lambda \cap \{(x,y), 0 \leq y \text{ and } x \leq g(y)\},$$

$$K_{2,\lambda}^{-} = K_\lambda \cap \{(x,y), y \leq 0 \text{ and } g(y) \leq x\}$$

and if we call:

v_λ^- the solution in $K_{1,\lambda}^-$,

w_λ^+ the solution in $K_{2,\lambda}^+$,

v_λ^+ the solution in $K_{1,\lambda}^+$,

w_λ^- the solution in $K_{2,\lambda}^-$, then we can put:

$$u_\lambda(x,y) = \begin{cases} v_\lambda^-(x,y) \text{ for } (x,y) \in K_{1,\lambda}^- \\ w_\lambda^+(x,y) \text{ for } (x,y) \in K_{2,\lambda}^+ \\ v_\lambda^+(x,y) \text{ for } (x,y) \in K_{1,\lambda}^+ \\ w_\lambda^-(x,y) \text{ for } (x,y) \in K_{2,\lambda}^-. \end{cases} \quad (6)$$

v_λ^- and v_λ^+ link up on γ because $v_\lambda^-(g(y,y)) = v_\lambda^+(g(y,y)) = \psi(y)$,

w_λ^- and w_λ^+ link up on γ because $w_\lambda^-(g(y,y)) = w_\lambda^+(g(y,y)) = \psi(y)$,

w_λ^- and v_λ^- link up on $(y=0)$ because $v_\lambda^-(x,0) = w_\lambda^-(x,0) = \varphi(x)$,

w_λ^+ and v_λ^+ link up on $(y=0)$ because $v_\lambda^+(x,0) = w_\lambda^+(x,0) = \varphi(x)$,

which assures the existence and the uniqueness of the solution u_λ in $K_\lambda = K_{1,\lambda}^- \cup K_{2,\lambda}^+ \cup K_{1,\lambda}^+ \cup K_{2,\lambda}^-$.

d) It remains to prove that the method actually gives a continuous global solution u in $\mathbb{R}^2$, that is, which verifies (P'_{int}).

If $\lambda_2 > \lambda_1$ then $K_{\lambda_1} \subset K_{\lambda_2}$; so, we must still prove that: $u_{\lambda_2|K_{\lambda_1}} = u_{\lambda_1}$.

But:

$$\forall (x,y) \in K_{\lambda_2},\ u_{\lambda_2}(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u_{\lambda_2}(\xi,\eta))d\xi d\eta$$

and we have this equality all the more for $(x,y) \in K_{\lambda_1}$. So we have:

$$u_{\lambda_2|K_{\lambda_1}}(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u_{\lambda_2|K_{\lambda_1}}(\xi,\eta))d\xi d\eta.$$

In other words: $u_{\lambda_2|K_{\lambda_1}}$ verifies (4) in K_{λ_1} and so coincides on it with its unique solution u_{λ_1}.

For every $(x,y) \in \mathbb{R}^2$, then we can put:

$$\begin{aligned} u(x,y) &= u_\lambda(x,y) \\ &= u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))d\xi d\eta \quad (7) \end{aligned}$$

where u_λ verifies (4) in K_λ and $(x, y) \in K_\lambda$.

The definition of u by (7) being independent of the compact K_λ, finally gives the unique global solution to (P'_{int}) or (P'_∞). $\square$

In chapter 6, we will need the estimations clarified by the following result.

2.2.1.2. Proposition

With the previous notations, for every compact $K \Subset \mathbb{R}^2$, there exits a compact $K_\lambda \Subset \mathbb{R}^2$, defined before, containing K, such that:

(i) $m_\lambda = \sup\limits_{(x,y)\in K_\lambda;\ t\in\mathbb{R}} \left|\frac{\partial F}{\partial z}(x, y, t)\right|$; $\Phi_\lambda = \|F(.,.,0)\|_{\infty,K_\lambda} + m_\lambda \|u_0\|_{\infty,K_\lambda}$;

(ii) $\|u\|_{\infty,K} \leq \|u\|_{\infty,K_\lambda} \leq \|u_0\|_{\infty,K_\lambda} + \dfrac{\Phi_\lambda}{m_\lambda} \exp[2\lambda' m_\lambda \lambda]$.

Proof.

We have:

$$u_n(x, y) = u_0(x, y) + \iint_{D(x,y,g)} F(\xi, \eta, u_{n-1}(\xi, \eta))d\xi d\eta, \quad n \geq 1,$$

$$u_0(x, y) = \psi(y) + \varphi(x) - \varphi(g(y)),$$

$$u_\lambda(x, y) = \begin{cases} v_\lambda^-(x, y) \text{ for } (x, y) \in K_{1,\lambda}^- \\ w_\lambda^+(x, y) \text{ for } (x, y) \in K_{2,\lambda}^+ \\ v_\lambda^+(x, y) \text{ for } (x, y) \in K_{1,\lambda}^+ \\ w_\lambda^-(x, y) \text{ for } (x, y) \in K_{2,\lambda}^-. \end{cases}$$

As:

$$\Phi_\lambda = \|F(.,.,0)\|_{\infty,K_\lambda} + m_\lambda \|u_{0,\varepsilon}\|_{\infty,K_\lambda},$$

$$V_n = u_n - u_{n-1},$$

according the proof of theorem 2.2.1.1., we have:

$$\forall n \in \mathbb{N}^*, \quad \|V_n\|_{\infty,K_{1,\lambda}^-} \leq m_\lambda^{n-1} \Phi_\lambda \frac{[(2\lambda')\lambda]^n}{n!} = \frac{\Phi_\lambda}{m_\lambda} \frac{[[2\lambda' m_\lambda \lambda]]^n}{n!}$$

and consequently:

$$\|u\|_{\infty,K_{1,\lambda}^-} \leq \|u_0\|_{\infty,K_{1,\lambda}^-} + \sum_{n=1}^{\infty} \|V_n\|_{\infty,K_{1,\lambda}^-} \leq \|u_0\|_{\infty,K_{1,\lambda}^-} + \frac{\Phi_\lambda}{m_\lambda} \exp[[2\lambda' m_\lambda \lambda]].$$

We can follow that:

$$\|u\|_{\infty,K_{1,\lambda}^-} \leq \|u_0\|_{\infty,K_{1,\lambda}^-} + \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda' m_\lambda \lambda]]$$

and just like it:

$$\|u\|_{\infty,K_{2,\lambda}^+} \leq \|u_0\|_{\infty,K_{2,\lambda}^+} + \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda' m_\lambda \lambda],$$
$$\|u\|_{\infty,K_{1,\lambda}^+} \leq \|u_0\|_{\infty,K_{1,\lambda}^+} + \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda' m_\lambda \lambda],$$
$$\|u\|_{\infty,K_{2,\lambda}^-} \leq \|u_0\|_{\infty,K_{2,\lambda}^-} + \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda' m_\lambda \lambda].$$

So:

$$\|u\|_{\infty,K_\lambda} \leq \|u_0\|_{\infty,K_\lambda} + \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda' m_\lambda \lambda],$$

hence:

$$\|u\|_{\infty,K} \leq \|u\|_{\infty,K_\lambda} \leq \|u_0\|_{\infty,K_\lambda} + \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda' m_\lambda \lambda].$$

□

Part II

Algebras of generalized functions

Chapter 3

The algebras of generalized functions

The algebras of generalized fonctions are the most effective tool to solve the non-linears differentials problems with irregular or characteristic data. To choose an appropriate structure for the considered Cauchy problem we use the results and notations of Marti [8], [9], [10] and [11].

3.1 The sheaves of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ algebras

3.1.1 Algebra structure

3.1.1.1. Notations

a) Let:

(1). Λ be a set of indixes;

(2). A be a subring of the ring $\mathbb{K}^{\Lambda}$, $(\mathbb{K} = \mathbb{R}$ ou $\mathbb{C})$;

(3). $A_+ = \{(r_\lambda)_\lambda \in A, r_\lambda \geq 0\}$;

(4). the following stability by overestimation property for A:

Whenever $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$ (i.e.: for each λ, $|s_\lambda| \leq r_\lambda$) for some $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times A_+$, it follows that: $(s_\lambda)_\lambda \in A$;

(5). I_A an ideal of A with the same property;

(6). a sheaf $\mathcal{E}$ of $\mathbb{K}$-topological algebras over a topological space X, such that, for each open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (p_i)_{i \in I(\Omega)}$ of semi-norms with the following property:

$$\forall i \in I(\Omega), \exists (j,k,C) \in I(\Omega) \times I(\Omega) \times \mathbb{R}_+^*, \forall f, g \in \mathcal{E}(\Omega) : p_i(fg) \leq C p_j(f) p_k(g),$$

(7). if Ω_1, Ω_2 are two open subsets of X with $\Omega_1 \subset \Omega_2$, it follows that $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$ then, for each $p_i \in \mathcal{P}(\Omega_1)$ the semi-norm $\widetilde{p}_i = p_i \circ \rho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$.

(8). Let $\mathcal{F} = (\Omega_h)_{h \in H}$ any family of open set in X and $\Omega = \underset{h \in H}{\cup} \Omega_h$. Then, for each $p_i \in \mathcal{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily of $\mathcal{F}$: $\Omega_1, \Omega_2, ..., \Omega_{n(i)}$ and corresponding semi-norms $p_1 \in \mathcal{P}(\Omega_1), p_2 \in \mathcal{P}(\Omega_2), ..., p_{n(i)} \in \mathcal{P}(\Omega_{n(i)})$, such that, for any $u \in \mathcal{E}(\Omega)$

$$p_i(u) \leq p_1\left(u_{|\Omega_1}\right) + p_2\left(u_{|\Omega_2}\right) + ... + p_{n(i)}\left(u_{|\Omega_{n(i)}}\right).$$

b) Then we put:

$$\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall i \in I(\Omega), ((p_i(u_\lambda))_\lambda \in A_+\}$$

$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) = \left\{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall i \in I(\Omega), (p_i(u_\lambda))_\lambda \in I_A^+\right\}$$

$$\mathcal{C} = A/I_A.$$

c) Remark: It is clear that A_+ is not a subring of A, but remains stable under addition and product. The same goes for I_A^+.

3.1.1.2. Proposition

If $|A| = \left\{ (|r_\lambda|)_\lambda \in \mathbb{R}_+^\Lambda : (r_\lambda)_\lambda \in A \right\}$ *and* $|I_A| = \left\{ (|r_\lambda|)_\lambda \in \mathbb{R}_+^\Lambda : (r_\lambda)_\lambda \in I_A \right\}$ *are respectively subsets of* A *and* I_A *it follows that* $|A| = A_+$ *and* $|I_A| = I_A^+$.

Proof. Obvious because we have evidently $A_+ \subset |A|$ and $I_A^+ \subset |I_A|$.

3.1.1.3. Proposition. See [12] and [13].

Under the above hypothesis, we obtain:

(i) $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ *is a sheaf of subalgebras of the sheaf* ε^Λ*;*

(ii) $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ *is a sheaf of ideals of* $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$.

Proof (Main steps).

We start from the statement that $\mathcal{E}$ and $\mathcal{E}^\Lambda$ are already sheaves of algebras. From (7), we can prove that $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ (and $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$) is a presheaf (the restriction property holds). The localization property does not require any hypothesis but, to glue the bits together, we need the property (8), which generalizes the situation from C^∞ to $\mathcal{E}$.

3.1.1.4. Proposition

Under the hypothesis of the above proposition, we obtain:

the constant sheaf $\mathcal{H}_{(A,\mathbb{K},|.|)}/\mathcal{J}_{(I_A,\mathbb{K},|.|)}$ *is exactly the sheaf* $\mathcal{C} = A/I_A$.

Proof. We clearly have $\mathcal{H}_{(A,\mathbb{K},|.|)} = A$ and $\mathcal{J}_{(I_A,\mathbb{K},|.|)} = I_A$.

3.1.1.5. Definition

We call sheaf of $(\mathcal{C},\mathcal{E},\mathcal{P})$*-algebra, the factor sheaf of algebras*

$$\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$$

and we denote by $[u_\lambda]$ *the class defined by the representative* $(u_\lambda)_{\lambda\in\Lambda}$.

3.1.1.6. Remark

In the context of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras, it is proved that if $A = A_+$, it follows that

$$\mathcal{H}_{(A,\mathbb{K},|.|)}/\mathcal{J}_{(I_A,\mathbb{K},|.|)} = A/I_A = \mathcal{C}.$$

But the first term is, in principle, a $(\mathcal{C}, \mathbb{K}, |.|)$-algebra and the second a ring of generalized constants, which is therefore an algebra. In fact, the following proposition will prove it:

3.1.1.7. Proposition

If A is a subring of $\mathbb{K}^\Lambda$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with the stability by overestimation, such that $|A| = A_+$, it follows that A is a $\mathbb{K}$-sub-algebra of $\mathbb{K}^\Lambda$.

Proof.

It suffices to show that A is stable under addition and product by elements of $\mathbb{K}$.

Let s be in $\mathbb{K}$ and $(r_\lambda)_\lambda \in A$. But $\mathbb{K}^\Lambda$ is a $\mathbb{K}$-vector space in which we have:

$$s.(r_\lambda)_\lambda = (s_\lambda)_\lambda (r_\lambda)_\lambda = (sr_\lambda)_\lambda .$$

But still, there is $n \in \mathbb{N}$ such that: $|s| \leq n$, so that:

$$|sr_\lambda| \leq |nr_\lambda| = |r_\lambda + ... + r_\lambda| = |\tau_\lambda|$$

where $\tau_\lambda \in A$.

So we have: $|sr_\lambda| \in |A|$, and, since $|A| = A_+$, we have: $s.(r_\lambda)_\lambda \in A$. □

3.1.2 Operations and properties.

3.1.2.1. Overgenerated rings.

In practice, the ring A and the ideal I_A are overgenerated by finite families of elements in accordance to the following definition :

Let

$$B_p = \left\{ (r_{n,\lambda})_\lambda \in (\mathbb{R}_+^*)^\Lambda, n = 1, 2, ..., p \right\}$$

and B be the subset of $(\mathbb{R}_+^*)^\Lambda$ obtained as products, quotients and linear combinations with coefficients in $\mathbb{R}_+^*$, of elements in B_p. Define :

$$A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists \, (b_\lambda)_\lambda \in B : |a_\lambda| \leq b_\lambda \right\}.$$

It is easy to see that A is a subring of $\mathbb{K}^\Lambda$ with the stability by overestimation property and moreover : $A_+ = |A|$. Then, we set the following definition:

3.1.2.2. Definition

In the previous situation, it is said that A is ovegenerated by B_p. If I_A is some ideal of A with the same stability by overestimation property, we can also say that $\mathcal{C} = A/I_A$ is overgenerated by B_p.

3.1.2.3. Example

As a “canonical” ideal of A, we can take:

$$I_A = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall \, (b_\lambda)_\lambda \in B : |a_\lambda| \leq b_\lambda \right\}.$$

3.1.2.4. The association process

We suppose that Λ is left-filtring for the given partial order relation $\prec$.

Let us denote by:

- Ω an open subset of X,
- E a given sheaf of topological $\mathbb{K}$-vector space containing $\mathcal{E}$ as a subsheaf,
- Φ a given application from Λ to $\mathbb{K}$ such that $(\Phi(\lambda))_\lambda = (\Phi_\lambda)_\lambda$ is an element of A.

We suppose also that we have:

$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \left\{ (u_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) : \lim_{\substack{E(\Omega) \\ \Lambda}} u_\lambda = 0 \right\}.$$

Then, for $u = [u_\lambda]$ and $v = [v_\lambda] \in \mathcal{E}(\Omega)$, we define the Φ-E association.

3.1.2.5. Definition

We denote by:

$$u \underset{E(\Omega)}{\overset{\Phi}{\approx}} v$$

the Φ-E association between u and v defined by:

$$\lim_{\substack{E(\Omega) \\ \Lambda}} \Phi_\lambda(u_\lambda - v_\lambda) = 0.$$

That is to say that for each neighbourhood V of 0 for the E-topology, there exits $\lambda_0 \in \Lambda$ such that:

$$\lambda \prec \lambda_0 \Longrightarrow \Phi_\lambda(u_\lambda - v_\lambda) \in V.$$

To ensure the independance of the definition with respect to the representatives of u and v, we must verify that if $\lim_{\substack{E(\Omega) \\ \Lambda}} \Phi_\lambda(w_\lambda) = 0$ holds, for some $(w_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$, then for any $(i_\lambda)_\lambda \in \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)$, $\lim_{\substack{E(\Omega) \\ \Lambda}} \Phi_\lambda(w_\lambda + i_\lambda) = 0$.

To prove the last condition, it is sufficient to show that:

$$(\Phi_\lambda i_\lambda)_\lambda \in \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega).$$

But for each $i \in I(\Omega)$, we have $p_i(\Phi_\lambda\,(i_\lambda)) = |\Phi_\lambda|\, p_i(i_\lambda)$. And, considering to the definitions and the stability properties given above, we have $|\Phi_\lambda|_\lambda \in A_+$ and $(p_i(i_\lambda))_\lambda \in I_A^+$. Then we also have $(|\Phi_\lambda|\, p_i(i_\lambda))_\lambda \in I_A^+$, which proves the required condition.

3.1.2.6. Remark

We can also define an association process between $u = [u_\lambda] \in E(\Omega)$ *and* $T \in E(\Omega)$ *by writing simply*

$$u \sim T \Longleftrightarrow \lim_{\substack{E(\Omega)\\ \Lambda}} u_\lambda = T.$$

Then taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda =]0,1]$, we come around to the association process defined in the literature [1] and [3].

3.1.2.7. Remark: Relation between unitary ring and injection

It is shown in [11] that a necessary and sufficient condition for the existence of a canonical sheaf morphism of algebra from $\mathcal{E}$ into $\mathcal{A}$ is that the ring A should be unitary.

If, in addition:

$$I_A \subset \left\{ (a_\lambda)_\lambda \in A : \lim_\Lambda a_\lambda = 0 \right\}$$

and, for each Ω, the $\mathcal{P}(\Omega)$ topology of $\mathcal{E}(\Omega)$ is separate, then this morphism is injective.

3.2 An adapted algebra to the generalized Cauchy problem

The first step is to link the problem and its data to algebraic and topological parameters that make it possible to build an appropriate ($\mathcal{C}$,$\mathcal{E}$,$\mathcal{P}$) algebra.

3.2.1 The sheaf $\mathcal{A}$

3.2.1.1. Definition

We choose $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and $\Lambda =]0,1]$. For all Ω, open set of $\mathbb{R}^d$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$ topology of the uniform convergence of all the derivatives on the compact subsets of Ω. This topology may be defined by the family of the semi-norms $P_{K,l}(u_\varepsilon) = \sup\limits_{|\alpha|\leq l} \left(\sup\limits_{x\in K} |D^\alpha u_\varepsilon(x)| \right)$ with $K \Subset \Omega$ and $D^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_d}}{\partial z_1^{\alpha_1}\ldots\partial z_d^{\alpha_d}}$ for $z = (z_1, ..., z_d) \in \Omega$, $l \in \mathbb{N}$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$.

We verify that it is compatible with the algebra structure of $\mathcal{E}(\Omega)$ since:

$$\forall K \Subset \Omega, \forall \alpha \in \mathbb{N}^d, \exists C > 0, \forall f, g \in \mathrm{C}^\infty(\Omega), P_{K,l}(fg) \leq P_{K,l}(f) P_{K,l}(g).$$

We put: $P_{K,\alpha}(u_\varepsilon) = \sup\limits_{x\in K} |D^\alpha u_\varepsilon(x)|$, so: $P_{K,l}(u_\varepsilon) = \sup\limits_{|\alpha|\leq l} (P_{K,\alpha}(u_\varepsilon))$.

We take $\Lambda =]0,1]$ and we index by ε instead of λ.

Let A be a subring of the ring $\mathbb{R}^\Lambda$ of family of reals with the usual laws. We consider an ideal I_A of A, A and I_A with the same estimation stability property. We suppose that $(1)_\varepsilon \in A$.

To simplify, we denote $\mathcal{X} = \mathcal{H}_{(A,\mathrm{C}^\infty,\mathcal{P})}$, $\mathcal{N} = \mathcal{J}_{(I_A,\mathrm{C}^\infty,\mathcal{P})}$ and $\mathcal{A} = \mathcal{X}/\mathcal{N}$.

We put:

$$\mathcal{X}(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\varepsilon))_\varepsilon \in A_+ \right\},$$

$$\mathcal{N}(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\varepsilon))_\varepsilon \in I_A^+ \right\}.$$

The ring of generalized constants associated with the factor algebra is exactly the factor ring $\mathcal{C} = A/I_A$. Finally, the generalized derivation $D^\alpha : u(= [u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$, provide $\mathcal{A}(\Omega)$ with a differential algebra structure.

3.2.1.2. Example

If we consider:

$$A = \mathbb{R}_M^\Lambda = \left\{ (m_\varepsilon)_\varepsilon \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}_+^*, \exists C \in \mathbb{R}_+^*, \exists \mu \in]0,1], \forall \varepsilon \in]0,\mu], |m_\varepsilon| \leq C\varepsilon^{-p} \right\}$$

and the ideal:

$$I_A = \left\{ (m_\varepsilon)_\varepsilon \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}_+^*, \exists D \in \mathbb{R}_+^*, \exists \mu \in]0,1], \forall \varepsilon \in]0,\mu], |m_\varepsilon| \leq D\varepsilon^{q} \right\},$$

then $\mathcal{A}\left(\mathbb{R}^d\right) = \mathcal{G}\left(\mathbb{R}^d\right)$ is the Colombeau generalized functions algebra.

3.2.2 Stability of $\mathcal{A}(\mathbb{R}^2)$ by an application

We extend the notation $F(.,.,u)$ to the case of u being a generalized function of the variable x, $x \in \mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$, in the following way:

3.2.2.1. Definition

Let Ω be an open set of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$. We say that the algebra $\mathcal{A}(\Omega)$ is ***stable*** *under F if we have the two following conditions:*

(i) For each $K \Subset \mathbb{R}^2$, for each $l \in \mathbb{N}$, for each $(u_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{]0,1]}$, there is a positive finite sequence C_1, C_2, ..., C_l, such that:

$$P_{K,l}(F(.,.,u_\varepsilon)) \leq \sum\nolimits_{i=0}^{l} C_i P_{K,l}^{i}(u_\varepsilon);$$

(ii) For each $K \Subset \mathbb{R}^2$, for each $l \in \mathbb{N}$, for each $(v_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$, $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$, there is a positive finite sequence D_1, D_2, ..., D_l, such that:

$$P_{K,l}(F(.,.,v_\varepsilon) - F\,(.,.,u_\varepsilon)) \leq \sum\nolimits_{j=0}^{l} D_j P_{K,l}^{j}(v_\varepsilon - u_\varepsilon).$$

3.2.2.2. Consequence

If $\mathcal{A}(\Omega)$ is stable under F then:

(i) For each $K \Subset \mathbb{R}^2$, for each $l \in \mathbb{N}$, for each $(u_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{]0,1]}$, we have:

$$\left(P_{K,l}(u_\varepsilon)\right)_\varepsilon \in A_+ \Longrightarrow \left(P_{K,l}(F(.,.,u_\varepsilon))\right)_\varepsilon \in A_+;$$

(ii) For each $K \Subset \mathbb{R}^2$, for each $l \in \mathbb{N}$, for each $(v_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$, $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$, we have:

$$\left(P_{K,l}(v_\varepsilon - u_\varepsilon)\right)_\varepsilon \in I_A^+ \Longrightarrow \left(P_{K,l}(F(.,.,v_\varepsilon) - F\,(.,.,u_\varepsilon))\right)_\varepsilon \in I_A^+.$$

3.2.2.3. Consequence

If $\mathcal{A}(\Omega)$ *is stable under* F *then, for each* $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$ *and for each* $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$*, we have*

$$(i) \qquad (F(.,.,u_\varepsilon))_\varepsilon \in \mathcal{X}(\Omega),$$

$$(ii) \qquad (F(.,.,u_\varepsilon + i_\varepsilon) - F(.,.,u_\varepsilon))_\varepsilon \in \mathcal{N}(\Omega).$$

3.2.2.4. Example

Let $F \in \mathrm{C}^\infty(\mathbb{R}^2, \mathbb{R})$ *definite by:* $F(x,y,z) = \dfrac{z}{1+z^2}$*, then* $\mathcal{A}\left(\mathbb{R}^2\right)$ *is stable under* F*.*

Proof:

We put:

$$f(z) = \frac{z}{1+z^2},$$

$$\Phi_\epsilon(x,y) = F(x,y,u_\varepsilon(x,y)) = \frac{u_\varepsilon(x,y)}{1+u_\varepsilon^2(x,y)}.$$

a) Study of f.

For each real z we have:

$$f(z) = \frac{z}{1+z^2} = \frac{i}{2}\left(\frac{1}{1+iz} - \frac{1}{1-iz}\right).$$

We put:

$$g_\alpha(z) = \frac{1}{1+\alpha z},$$

with: $\alpha = i$ or $\alpha = -i$.

Let us show by induction, that for each integer $n \geq 1$, we have:

$$g_\alpha^{(n)}(z) = \frac{(-1)^n (n!)\, \alpha^n}{(1+\alpha z)^{n+1}}.$$

We have:

$$g_{\alpha}'(z) = \frac{-\alpha}{(1+\alpha z)^2}.$$

Suppose that:

$$g_{\alpha}^{(n)}(z) = \frac{(-1)^n (n!) \alpha^n}{(1+\alpha z)^{n+1}},$$

then:

$$g_{\alpha}^{(n+1)}(z) = (-1)^n (n!) \alpha^n \left[\frac{-(n+1)(1+\alpha z)^n \alpha}{(1+\alpha z)^{2n+2}} \right] = \frac{(-1)^{n+1} ((n+1)!) \alpha^{n+1}}{(1+\alpha z)^{n+2}},$$

therefore, by the principle of induction, the property is true for each $n \geq 1$.

We have:

$$f^{(n)}(z) = \frac{i}{2} \left(g_i^{(n)}(z) - g_{-i}^{(n)}(z) \right),$$

and, for $\alpha = i$ or $\alpha = -i$, we have:

$$\left| g_{\alpha}^{(n)}(z) \right| \leq \left| \frac{(-1)^n (n!) \alpha^n}{(1+\alpha z)^{n+1}} \right| \leq (n!) \frac{|i|^n}{(1+z^2)^{n+1}} \leq n!,$$

so:

$$\left| f^{(n)}(z) \right| \leq \frac{1}{2} \left(\left| g_i^{(n)}(z) \right| + \left| g_{-i}^{(n)}(z) \right| \right) \leq n!.$$

All the sucessive derivatives of f are therefore bounded on $\mathbb{R}$, and for each integer n:

$$\sup_{z \in \mathbb{R}} \left| f^{(n)}(z) \right| \leq n!.$$

b) Let us show that for each n, there is $C_{r,n} > 0$, $1 \leq r \leq n$, such that we have:

$$P_{K,n}(F(.,.,u_\varepsilon)) \leq \sum\nolimits_{r=1}^{r=n} C_{r,n} P_{K,n}^r (u_\varepsilon).$$

In the expression of $\Phi_\epsilon(x,y) = F(x,y,u_\varepsilon(x,y))$, x and y have similar roles therefore the study of $\dfrac{\partial^n \Phi_\varepsilon}{\partial x^k \partial y^{n-k}}$ is similar to these of $\dfrac{\partial^n \Phi_\varepsilon}{\partial x^{n-k} \partial y^k}$. Then we can prove the relation only for $\dfrac{\partial^n \Phi_\varepsilon}{\partial x^n}$.

b1) Overestimation of $P_{K,1}\left(F\left(.,.,u_{\varepsilon}\right)\right)$.

We have:

$$\frac{\partial \Phi_{\varepsilon}}{\partial x}\left(x,y\right)=f'(u_{\varepsilon}\left(x,y\right))\frac{\partial u_{\varepsilon}}{\partial x}\left(x,y\right),$$

hence:

$$\forall K\Subset\mathbb{R}^{2},P_{K,(1,0)}\left(F\left(.,.,u_{\varepsilon}\right)\right)\leq P_{K,(1,0)}\left(u_{\varepsilon}\right).$$

Consequently:

$$\forall K\Subset\mathbb{R}^{2},P_{K,1}\left(F\left(.,.,u_{\varepsilon}\right)\right)\leq P_{K,1}\left(u_{\varepsilon}\right).$$

b2) Overestimation of $P_{K,2}\left(F\left(.,.,u_{\varepsilon}\right)\right)$.

For each $K\Subset\mathbb{R}^{2}$, we have:

$$\frac{\partial^{2}\Phi_{\varepsilon}}{\partial x\partial y}\left(x,y\right)=f^{(2)}(u_{\varepsilon}\left(x,y\right))\frac{\partial u_{\varepsilon}}{\partial y}\left(x,y\right)\frac{\partial u_{\varepsilon}}{\partial x}\left(x,y\right)+f'(u_{\varepsilon}\left(x,y\right))\frac{\partial^{2}u_{\varepsilon}}{\partial x\partial y}\left(x,y\right),$$

hence:

$$P_{K,(1,1)}\left(F\left(.,.,u_{\varepsilon}\right)\right)\leq 2P_{K,1}^{2}\left(u_{\varepsilon}\right)+P_{K,2}\left(u_{\varepsilon}\right)\leq 2P_{K,2}^{2}\left(u_{\varepsilon}\right)+P_{K,2}\left(u_{\varepsilon}\right).$$

We have:

$$\frac{\partial^{2}\Phi_{\varepsilon}}{\partial x^{2}}\left(x,y\right)=f^{(2)}(u_{\varepsilon}\left(x,y\right))\left(\frac{\partial u_{\varepsilon}}{\partial x}\right)^{2}\left(x,y\right)+f'(u_{\varepsilon}\left(x,y\right))\frac{\partial^{2}u_{\varepsilon}}{\partial x^{2}}\left(x,y\right),$$

then:

$$P_{K,(2,0)}\left(F\left(.,.,u_{\varepsilon}\right)\right)\leq 2P_{K,1}^{2}\left(u_{\varepsilon}\right)+P_{K,2}\left(u_{\varepsilon}\right)\leq 2P_{K,2}^{2}\left(u_{\varepsilon}\right)+P_{K,2}\left(u_{\varepsilon}\right).$$

Consequently:

$$\forall K\Subset\mathbb{R}^{2},\ P_{K,2}\left(F\left(.,.,u_{\varepsilon}\right)\right)\leq 2P_{K,2}^{2}\left(u_{\varepsilon}\right)+P_{K,2}\left(u_{\varepsilon}\right).$$

b3) Overestimation of $P_{K,3}\left(F\left(.,.,u_\varepsilon\right)\right)$.

For each $K \Subset \mathbb{R}^2$, we have:

$$\frac{\partial^3 \Phi_\varepsilon}{\partial x^2 \partial y}\left(x,y\right) = f^{(3)}(u_\varepsilon\left(x,y\right))\frac{\partial u_\varepsilon}{\partial y}\left(x,y\right)\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\left(x,y\right) + f^{(2)}(u_\varepsilon\left(x,y\right))\frac{\partial u_\varepsilon}{\partial y}\left(x,y\right)\frac{\partial^2 u_\varepsilon}{\partial x^2}\left(x,y\right)$$

$$+ 2f^{(2)}(u_\varepsilon\left(x,y\right))\frac{\partial u_\varepsilon}{\partial x}\left(x,y\right)\frac{\partial^2 u_\varepsilon}{\partial x \partial y}\left(x,y\right) + f'(u_\varepsilon\left(x,y\right))\frac{\partial^3 u_\varepsilon}{\partial^2 x \partial y}\left(x,y\right),$$

hence:

$$P_{K,(2,1)}\left(F\left(.,.,u_\varepsilon\right)\right) \leq 3!P_{K,1}^3\left(u_\varepsilon\right) + 3.2!P_{K,1}\left(u_\varepsilon\right)P_{K,2}\left(u_\varepsilon\right) + P_{K,3}\left(u_\varepsilon\right).$$

We have:

$$\frac{\partial^3 \Phi_\varepsilon}{\partial x^3}\left(x,y\right) = f^{(3)}(u_\varepsilon\left(x,y\right))\left(\frac{\partial u_\varepsilon}{\partial x}\right)^3\left(x,y\right) + 3f^{(2)}(u_\varepsilon\left(x,y\right))\frac{\partial u_\varepsilon}{\partial x}\left(x,y\right)\frac{\partial^2 u_\varepsilon}{\partial x^2}\left(x,y\right)$$

$$+ f'(u_\varepsilon\left(x,y\right))\frac{\partial^3 u_\varepsilon}{\partial x^3}\left(x,y\right),$$

then:

$$P_{K,(3,0)}\left(F\left(.,.,u_\varepsilon\right)\right) \leq 6P_{K,1}^3\left(u_\varepsilon\right) + 3.2!P_{K,1}\left(u_\varepsilon\right)P_{K,2}\left(u_\varepsilon\right) + 1P_{K,3}\left(u_\varepsilon\right).$$

Consequently, for each $K \Subset \mathbb{R}^2$:

$$\begin{aligned} P_{K,3}\left(F\left(.,.,u_\varepsilon\right)\right) &\leq 6P_{K,1}^3\left(u_\varepsilon\right) + 6P_{K,1}\left(u_\varepsilon\right)P_{K,2}\left(u_\varepsilon\right) + P_{K,3}\left(u_\varepsilon\right) \\ &\leq 6P_{K,3}^3\left(u_\varepsilon\right) + 6P_{K,3}^2\left(u_\varepsilon\right) + P_{K,3}\left(u_\varepsilon\right). \end{aligned}$$

b4) *Lemma:*

The n-th derivative, $(f \circ u)^{(n)}$, *of* $f \circ u$ *can be written:*

$$(f \circ u)^{(n)} = \sum_{r=1}^{n} \sum_{\substack{i_1 \geq \ldots \geq i_r \\ i_1 + \ldots + i_r = n}} t_{i_1,\ldots,i_r} f^{(r)} \circ u . \prod_{k=1}^{r} u^{(i_k)}$$

where the coefficients $t_{i_1,\ldots,i_r}$ *are integers.*

Therefore we have, for $\alpha = n$ and $\beta = 0$:

$$\frac{\partial^n \Phi_\varepsilon}{\partial x^n}(x,y) = \sum_{r=1}^{n} \sum_{\substack{i_1 \geq \ldots \geq i_r \\ i_1 + \ldots + i_r = n}} (t_{i_1,\ldots,i_r})\, f^{(r)}(u_\varepsilon(x,y)) \prod_{k=1}^{r} \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y),$$

For each $K \Subset \mathbb{R}^2$, for each $i_k \in \mathbb{N}$, $i_k \leq n$, for each $r \in \mathbb{N}$,

$$\sup_{(x,y)\in K} \left| f^{(r)}(u_\varepsilon(x,y)) \right| \leq r! \leq n!,$$

therefore:

$$\max_{1 \leq i_k \leq n} \left(\sup_{(x,y)\in K} \left| f^{(i_k)}(u_\varepsilon(x,y)) \right| \right) \leq n!.$$

We have:

$$\sup_{(x,y)\in K} \left| \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y) \right| \leq P_{K,i_k}(u_\varepsilon) \leq P_{K,n}(u_\varepsilon)$$

and:

$$\sup_{(x,y)\in K} \left(\left| \prod_{k=1}^{r} \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y) \right| \right) \leq P_{K,n}^{r}(u_\varepsilon),$$

therefore:

$$\sup_{(x,y)\in K} \left| (t_{i_1,\ldots,i_r})\, f^{(r)}(u_\varepsilon(x,y)) \prod_{k=1}^{r} \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y) \right| \leq (t_{i_1,\ldots,i_r})\, n! P_{K,n}^{r}(u_\varepsilon).$$

Consequently:

$$\sup_{(x,y)\in K} \left| \frac{\partial^n \Phi_\varepsilon}{\partial x^n}(x,y) \right| \leq \sum_{r=1}^{n} \left(\sum_{\substack{i_1 \geq \ldots \geq i_r \\ i_1 + \ldots + i_r = n}} (t_{i_1,\ldots,i_r}) \right) n! P_{K,n}^{r}(u_\varepsilon).$$

c) Let us show that: for each $K \Subset \mathbb{R}^2$, for each $l \in \mathbb{N}$, for each $(v_\varepsilon)_\varepsilon$, $(u_\varepsilon)_\varepsilon$ elements of $\mathcal{X}(\Omega)$, there is a positive number D_l, such that:

$$P_{K,l}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \leq D_l P_{K,l}(v_\varepsilon - u_\varepsilon).$$

c1) First let us show the relation for $l = 1$.

For each $K \Subset \mathbb{R}^2$, for each $(x, y) \in (x, y)$, we have:

$$\begin{aligned} g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y)) &= \frac{1}{1+\alpha v_\varepsilon(x,y)} - \frac{1}{1+\alpha u_\varepsilon(x,y)} \\ &= \frac{\alpha\left(u_\varepsilon(x,y) - v_\varepsilon(x,y)\right)}{\left(1+\alpha v_\varepsilon(x,y)\right)\left(1+\alpha u_\varepsilon(x,y)\right)}, \end{aligned}$$

so:

$$\begin{aligned} |g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y))| &\leq \frac{|\alpha|\,|u_\varepsilon(x,y) - v_\varepsilon(x,y)|}{|1+\alpha v_\varepsilon(x,y)|\,|1+\alpha u_\varepsilon(x,y)|} \leq \frac{|u_\varepsilon(x,y) - v_\varepsilon(x,y)|}{|1+v_\varepsilon^2(x,y)|\,|1+u_\varepsilon^2(x,y)|} \\ &\leq |v_\varepsilon(x,y) - u_\varepsilon(x,y)|, \end{aligned}$$

because $\alpha = i$ or $\alpha = -i$.

Since:

$$f(z) = \frac{z}{1+z^2} = \frac{i}{2}\left(g_i(z) - g_{-i}(z)\right),$$

then:

$$f(v_\varepsilon(x,y)) - f(u_\varepsilon(x,y)) = \frac{i}{2}\left[\left(g_i(v_\varepsilon(x,y)) - g_i(u_\varepsilon(x,y)) - \left(g_{-i}(v_\varepsilon(x,y)) - g_{-i}(u_\varepsilon(x,y))\right)\right)\right]$$

and:

$$\begin{aligned} |f(v_\varepsilon(x,y)) - f(u_\varepsilon(x,y))| &\leq \frac{1}{2}\left[\left|g_i(v_\varepsilon(x,y)) - g_i(u_\varepsilon(x,y))\right| + \left|g_{-i}(v_\varepsilon(x,y)) - g_{-i}(u_\varepsilon(x,y))\right|\right] \\ &\leq |v_\varepsilon(x,y) - u_\varepsilon(x,y)|, \end{aligned}$$

consequently:

$$P_{K,0}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \leq P_{K,0}(v_\varepsilon - u_\varepsilon).$$

c2) To establish the relation for g_α is enough

For each $K \Subset \mathbb{R}^2$, for each $(x,y) \in (x,y)$, we have:

$$\Psi_\varepsilon(x,y) = g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y)) = \frac{-\alpha}{(1+\alpha v_\varepsilon(x,y))(1+\alpha u_\varepsilon(x,y))}(v_\varepsilon(x,y) - u_\varepsilon(x,y))$$

and:

$$|\Psi_\varepsilon(x,y)| \leq |g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y))| \leq |v_\varepsilon(x,y) - u_\varepsilon(x,y)|,$$

so:

$$\sup_{(x,y)\in K} |\Psi_\varepsilon(x,y)| \leq P_{K,0}(v_\varepsilon - u_\varepsilon).$$

We put:

$$h_\varepsilon(x,y) = \frac{-\alpha}{(1+\alpha v_\varepsilon(x,y))(1+\alpha u_\varepsilon(x,y))} = -\alpha g_\alpha(v_\varepsilon(x,y)) g_\alpha(u_\varepsilon(x,y)).$$

Since g_α and all the sucessive derivatives are bounded, for each integer n, $\dfrac{\partial^n h_\varepsilon}{\partial x^n}$ is bounded on K by a polynomial of $\|v_\varepsilon\|_{\infty,K}$, $\|u_\varepsilon\|_{\infty,K}$, $\left\|\dfrac{\partial v_\varepsilon}{\partial x}\right\|_{\infty,K}$, $\left\|\dfrac{\partial u_\varepsilon}{\partial x}\right\|_{\infty,K}$, ..., $\left\|\dfrac{\partial^n v_\varepsilon}{\partial x^n}\right\|_{\infty,K}$, $\left\|\dfrac{\partial^n u_\varepsilon}{\partial x^n}\right\|_{\infty,K}$, with positive coefficients, what we write $d_n(K, u_\varepsilon, v_\varepsilon)$.

According to Leibniz's formula for the successive derivatives of a product, we have:

$$\frac{\partial^n \Psi_\varepsilon}{\partial x^n}(x,y) = -\alpha \sum_{i=0}^{i=n} C_n^i \frac{\partial^i h_\varepsilon}{\partial x^i}(x,y) \frac{\partial^{n-i}(v_\varepsilon - u_\varepsilon)}{\partial x^{n-i}}(x,y).$$

Consequently:

$$\begin{aligned}\sup_{(x,y)\in K)} \left|\frac{\partial^n \Psi_\varepsilon}{\partial x^n}(x,y)\right| &\leq \sum_{i=0}^{i=n} C_n^i d_i(K, u_\varepsilon, v_\varepsilon) P_{K,n-i}(v_\varepsilon - u_\varepsilon) \\ &\leq \left(\sum_{i=0}^{i=n} C_n^i d_i(K, u_\varepsilon, v_\varepsilon)\right) P_{K,n}(v_\varepsilon - u_\varepsilon).\end{aligned}$$

From this, it may be deduced that:

$$P_{K,n}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \leq D_n P_{K,n}(v_\varepsilon - u_\varepsilon).$$

□

3.3 Singular parametric spectrum

3.3.1 Analysis of the distribution singularities of a generalized function

3.3.1.1. Notations

We suppose that:

$$\mathcal{N}_{\mathcal{D}'}^{\mathcal{A}}(\Omega) = \left\{(u_\varepsilon) \in \mathcal{X}(\Omega), \lim_{\varepsilon\to 0} u_\varepsilon = 0, \text{ in } \mathcal{D}'(\Omega)\right\} \supset \mathcal{N}(\Omega).$$

Then we put:

$$\mathcal{D}'_{\mathcal{A}}(\Omega) = \left\{[u_\varepsilon] \in \mathcal{A}(\Omega), \exists T \in \mathcal{D}'(\Omega) \lim_{\varepsilon\to 0} (u_\varepsilon) = T, \text{ in } \mathcal{D}'(\Omega)\right\}.$$

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is clearly defined because the limit is independent from the chosen representative; indeed:

$$\lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\Omega)}} (u_\varepsilon + i_\varepsilon) = \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\Omega)}} u_\varepsilon + \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\Omega)}} i_\varepsilon = \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\Omega)}} u_\varepsilon \text{ puisque : } \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\Omega)}} i_\varepsilon = 0.$$

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ appears like a $\mathbb{R}$-vector subspace of $\mathcal{A}(\Omega)$.

Therefore we can consider $\mathcal{O}_{D'_{\mathcal{A}}}$ “the set of all x having a neighbourhood V on which u is associated to a distribution”:

$$\mathcal{O}_{D'_{\mathcal{A}}}(u) = \left\{x \in \Omega / \exists V \in \mathcal{V}(x) : u\,|_V \in \mathcal{D}'_{\mathcal{A}}(V)\right\},$$

$\mathcal{V}(x)$ being the set of all neighbourhood of x.

3.3.1.2. Definition

The $\mathcal{D}'$-singular support of $u \in \mathcal{A}(\Omega)$ is denoted singsupp$_{\mathcal{D}'}(u) = S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$ and defined as $S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) = \Omega \backslash O_{\mathcal{D}'_{\mathcal{A}}}(u)$.

3.3.2 Elements of parametric microlocal analysis

Let $u \in \mathcal{A}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. It may happen that $u = [u_\varepsilon]$ is not associated with any distribution in a neighbourhood of x, that is, there is no open neighbourhood V_x of x for which $\lim\limits_{\varepsilon \to 0}(u_{\varepsilon|V_x})$ belongs to $\mathcal{D}'(V_x)$. [13]

But in this case, it may happen that some real number r and some neighbourhood V_x of x exist such that $\lim\limits_{\varepsilon \to 0}(\varepsilon^r u_{\varepsilon|V_x})$ belongs to $\mathcal{D}'(V_x)$, that is, $[\varepsilon^r u_\varepsilon]$ belongs to $\mathcal{D}'_{\mathcal{A}}(V_x)$, the vector subspace of $\mathcal{A}(V_x)$ whose elements u are associated with some distribution of $\mathcal{D}'(V_x)$.

For example, let us take $\varphi \in \mathcal{D}(\mathbb{R})$, $\varphi \geq 0$, $\int \varphi(x)dx = 1$ and $u_\varepsilon(x) = \varepsilon^{-2}\varphi(x\varepsilon^{-1})$. Then, $u = [u_\varepsilon]$ is a generalized function of $\mathcal{A}(\mathbb{R})$ which is not associated with a distribution in a neighbourhood of 0, but for $r \geq 1$, $[\varepsilon^r u_\varepsilon]$ is a distribution.

This leads to the following concept:

3.3.3 Singular parametric spectrum

3.3.3.1. Notations

Let Ω be an open set of $\mathbb{R}^d$. For $x \in \Omega$ and $u = [u_\varepsilon] \in \mathcal{A}(\Omega)$, we put:

$$N_{\mathcal{D}',x}(u) = \left\{ r \in \mathbb{R}_+ ; \exists V_x \in \mathcal{V}(x) : \lim_{\varepsilon \to 0}(\varepsilon^r u_{\varepsilon|V_x}) \in \mathcal{D}'(V_x) \right\}$$

we can show that $N_{\mathcal{D}',x}(u)$ does not depend on the chosen representative of u and that if $N_{\mathcal{D}',x}(u)$ contains some $r_0 \in \mathbb{R}_+$, it must contain every r, $r \geq r_0$.

Then one defines the $\mathcal{D}'$-fiber over x as: $\Sigma_{\mathcal{D}',x}(u) = \mathbb{R}_+ \backslash N_{\mathcal{D}',x}(u)$.

This is either a bounded interval of $\mathbb{R}_+$ with the form $[0, r[$ or $[0, r]$, either $\mathbb{R}_+$ itself, or the empty set.

Then we can give the following definition of the singular parametric spectrum of generalized function:

3.3.3.2. Definition

We define the $\mathcal{D}'$-singular parametric spectrum of $u \in \mathcal{A}(\Omega)$ *as the subset of* $\Omega \times \mathbb{R}_+$*:*

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u = \left\{ (x,r) \in \Omega \times \mathbb{R}_+, r \in \Sigma_{\mathcal{D}',x}(u) \right\}.$$

3.3.3.3. Remark

We have $\Sigma_{\mathcal{D}',x}(u) = \varnothing$ if and only if there exists a neighbourhood V_x of x such that:

$$\lim_{\varepsilon \to 0} (u_{\varepsilon|V_x}) \in \mathcal{D}'(V_x),$$

that is, if and only if x does not belong to the $\mathcal{D}'$-singular support of u: $\mathcal{S}^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$.

It results from this that the projection on Ω of $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u$ is exactly $S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u$.

3.3.4 Some properties of the $\mathcal{D}'$-singular parametric spectrum $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u$ of a generalized function $u \in \mathcal{A}(\Omega)$

3.3.4.1. Theorem

Let u *and* $v \in \mathcal{A}(\Omega)$*. Then we have:*

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} (u+v) = S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} (u) \cup S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} (v).$$

Proof.

Let $r \in N_{\mathcal{D}',x}(u) \cap N_{\mathcal{D}',x}(v)$, then, there exist $V_x \in \mathcal{V}(x)$ and $W_x \in \mathcal{V}(x)$ such that:

$$\lim_{\varepsilon \to 0} (\varepsilon^r u_{\varepsilon|V_x}) \in \mathcal{D}'(V_x) \text{ et } \lim_{\varepsilon \to 0} (\varepsilon^r u_{\varepsilon|W_x}) \in \mathcal{D}'(W_x).$$

From this it may be deduced that:

$$\lim_{\varepsilon \to 0} (\varepsilon^r \left(u_\varepsilon + v_\varepsilon \right)_{|V_x \cap W_x}) \in \mathcal{D}'(V_x \cap W_\varepsilon),$$

it follows that:

$$r \in N_{\mathcal{D}',x}(u+v),$$

consequently:

$$N_{\mathcal{D}',x}(u) \cap N_{\mathcal{D}',x}(v) \subset N_{\mathcal{D}',x}(u+v).$$

We obtain the result by taking the complementary sets in $\mathbb{R}_+$. □

3.3.4.2. Corollary

For any u, u_0, u_1 *in* $\mathcal{A}(\Omega)$ *with:*

(i) $u = u_0 + u_1$, (ii) $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u_0) = \varnothing$,

we have:

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) = S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u_1).$$

Proof.

The previous theorem and the condition (ii) give:

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u_1).$$

But, as (i) implies:

$$u_0 = u - u_1,$$

we obtain of course the converse inclusion, and thus the result. □

3.3.4.3. Theorem

Let $u \in \mathcal{A}(\Omega)$. *Then we have, for each* D^α, $\alpha \in \mathbb{N}^d$

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(D^\alpha u) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u).$$

Proof.

Let $r \in N_{\mathcal{D}',x}(u)$. There exists $V_x \in \mathcal{V}(x)$ such that: $\lim\limits_{\varepsilon \to 0}(\varepsilon^r u_{\varepsilon|V_x}) = T \in \mathcal{D}'(V_x)$.

The continuity of D^α implies that:

$\lim\limits_{\varepsilon \to 0}(\varepsilon^r D^\alpha u_{\varepsilon|V_x}) = \lim\limits_{\varepsilon \to 0} D^\alpha(\varepsilon^r u_{\varepsilon|V_x}) = D^\alpha T \in \mathcal{D}'(V_x)$.

Thus: $N_{\mathcal{D}',x}(u) \subset N_{\mathcal{D}',x}(D^\alpha u)$; We obtain the result by taking the complementary sets in $\mathbb{R}_+$. □

3.3.4.4. Theorem

Let $f \in \mathrm{C}^\infty(\Omega)$ and $u \in \mathcal{A}(\Omega)$. *Then we have:*

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(fu) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u).$$

Proof. Let $r \in N_{\mathcal{D}',x}(u)$. There exists $V_x \in \mathcal{V}(x)$ such that: $\lim\limits_{\varepsilon \to 0}(\varepsilon^r u_{\varepsilon|V_x}) = T \in \mathcal{D}'(V_x)$ that is, for each $\varphi \in \mathcal{D}(V_x)$, we have:

$$\lim_{\varepsilon \to 0} \int \varepsilon^r u_\varepsilon(x)\,\varphi(x)\,dx = \langle T, \varphi \rangle.$$

Thus, we have:

$$\lim_{\varepsilon \to 0} \int \varepsilon^r (f u_\varepsilon)(x)\,\varphi(x)\,dx = \lim_{\varepsilon \to 0} \int \varepsilon^r u_\varepsilon(x)\,(f\varphi)(x)\,dx = \langle T, f\varphi \rangle = \langle fT, \varphi \rangle,$$

it follows that:

$$\lim_{\varepsilon \to 0}(\varepsilon^r f u_{\varepsilon|V_x}) = fT \in \mathcal{D}'(V_x),$$

therefore r belongs to $N_{\mathcal{D}',x}(fu)$.

From the estimation

$$N_{\mathcal{D}',x}(u) \subset N_{\mathcal{D}',x}(fu),$$

we can deduce the result. □

3.3.4.5. Corollary

Let $P(D)=\sum_{|\alpha|\leq m} C_\alpha D^\alpha$ *be a differential polynomial with coefficients in* $\mathrm{C}^\infty(\Omega)$.

Then, for any $u \in \mathcal{A}(\Omega)$ *we have:*

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}\left(P(D)u\right) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}\left(u\right).$$

Proof. We can write

$$P(D)u=\sum_{|\alpha|\leq m} C_\alpha D^\alpha u$$

and apply the previous theorems. □

Part III

Generalized problems

Chapter 4

A generalized Cauchy problem

4.1 Terms of the problem

4.1.1 Problem (P_G)

We take up again the formulation of the Cauchy problem posed in 1.1.1. under the form:

$$(P_G)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|\gamma} = \varphi \\ \dfrac{\partial u}{\partial y} \,|_{\gamma} = \psi, \end{cases}$$

where $\varphi = [\varphi_\varepsilon]$ and $\psi = [\psi_\varepsilon]$ and the hypothesis on F, f, φ_ε, ψ_ε are kept but, now u is searched in an algebra of generalized functions $\mathcal{A}\left(\mathbb{R}^2\right)$ defined in the previous chapter.

We suppose that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under F, that $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}\left(\mathbb{R}^2\right)$ are built so on the same ring of generalized constants.

We suppose that, for every ε, the problems:

$$P_\infty\left(\varphi_\varepsilon, \psi_\varepsilon\right)\begin{cases} \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y}(x,y) = F(x,y,u_\varepsilon(x,y)) \\ u_\varepsilon\left(x, f(x)\right) = \varphi_\varepsilon(x) \\ \dfrac{\partial u_\varepsilon}{\partial y}\left(x, f(x)\right) = \psi_\varepsilon(x) \end{cases}$$

have a solution $u_\epsilon \in \mathrm{C}^\infty(\mathbb{R}^2)$.

4.1.2 Giving a meaning to (P_G)

Giving a meaning to (P_G) is first giving a meaning to:

$$\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) & (1_G) \\ u_{|\gamma} = \varphi \in \mathcal{A}(\mathbb{R}) & (2_G) \\ \dfrac{\partial u}{\partial y} \,|_\gamma = \psi \in \mathcal{A}(\mathbb{R}) & (3_G) \end{cases}$$

when $u \in \mathcal{A}\left(\mathbb{R}^2\right)$ and γ is the smooth submanifold of $\mathbb{R}^2$ defined by $y = f(x)$.

Giving a meaning to (1_G), under the hypothesis that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable by F, signifies that for a representative $(u_\varepsilon)_\varepsilon$ of u we must have, for every $(i_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$ and $(j_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$,

$$\left(\frac{\partial^2(u_\varepsilon + i_\varepsilon)}{\partial x \partial y} - F(.,.,u_\varepsilon + j_\varepsilon)\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right).$$

As: $\left(\dfrac{\partial^2(u_\varepsilon + i_\varepsilon)}{\partial x \partial y} - \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y}\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$ and since: $(F(.,.,u_\varepsilon + j_\varepsilon) - F(.,.,u_\varepsilon))_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$, this comes down to verifying that:

$$\underline{\left(\frac{\partial^2(u_\varepsilon)}{\partial x \partial y} - F(.,.,u_\varepsilon)\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right).}$$

Giving a meaning to (2_G) and (3_G) signifies first defining $u_{|\gamma}$ and $\dfrac{\partial u}{\partial y}_{|\gamma}$ and, as γ is a smooth submanifold of $\mathbb{R}^2$ that can be represented by a single map ($\gamma = f(x)$), we can identify $\mathcal{A}(\gamma)$ and $\mathcal{A}(\mathbb{R})$ and so $u_{|\gamma}$ to the element of $\mathcal{A}(\mathbb{R})$ a representative of which is $(x \mapsto u_\varepsilon(x, f(x))_\varepsilon$ and we can identify $\dfrac{\partial u}{\partial y}_{|\gamma}$ to the element of $\mathcal{A}(\mathbb{R})$ a representative of which is $\left(x \mapsto \dfrac{\partial u_\varepsilon}{\partial y}(x, f(x))\right)_\varepsilon$.

So (2_G) is equivalent to:

$(x \mapsto ((u_\varepsilon + i_\varepsilon)(x, f(x)) - (\varphi_\varepsilon + \alpha_\varepsilon)(x))_\varepsilon \in \mathcal{N}(\mathbb{R})$.

(3_G) is equivalent to:

$$(x \mapsto ((\frac{\partial u_\varepsilon + i_\varepsilon}{\partial y})(x, f(x)) - (\psi_\varepsilon + \beta_\varepsilon)(x))_\varepsilon \in \mathcal{N}(\mathbb{R})$$

for every $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$, $(\alpha_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$, $(\beta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$, and considering:

$$(x \mapsto ((u_\varepsilon + i_\varepsilon)(x, f(x)) - u_\varepsilon(x, f(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(x \mapsto ((\varphi_\varepsilon + \alpha_\varepsilon)(x) - \varphi_\varepsilon(x))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(x \mapsto ((\frac{\partial u_\varepsilon + i_\varepsilon}{\partial y})(x, f(x)) - \frac{\partial u_\varepsilon}{\partial y}(x, f(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(x \mapsto ((\psi_\varepsilon + \beta_\varepsilon)(x) - \psi_\varepsilon(x))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(x \mapsto (j_\varepsilon(x) - i_\varepsilon(x, f(x))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

this comes down to:

$$\underline{(x \mapsto (u_\varepsilon(x, f(x)) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R})},$$

$$\underline{(x \mapsto (\frac{\partial u_\varepsilon}{\partial y}(x, f(x)) - \psi_\varepsilon(x))_\varepsilon \in \mathcal{N}(\mathbb{R}).}$$

To sum up, (P_G) has a meaning if and only if it is represented by a $(u_\varepsilon)_\varepsilon$verifying:

$$\begin{cases} \left(\dfrac{\partial^2 (u_\varepsilon)}{\partial x \partial y} - F(.,.,u_\varepsilon)\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right) \\ (x \mapsto (u_\varepsilon(x, f(x)) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}) \\ (x \mapsto (\dfrac{\partial u_\varepsilon}{\partial y}(x, f(x)) - \psi_\varepsilon(x))_\varepsilon \in \mathcal{N}(\mathbb{R}). \end{cases}$$

If so, for every ε, the solution u_ε to $P_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ is such that $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$ then the relations above are all the more true and $[u_\varepsilon]$ is a solution to (P_G). □

4.2 Solving the problem

4.2.1 Solving (P_G)

4.2.1.1. Theorem

Let us suppose that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under F, let us suppose that $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}\left(\mathbb{R}^2\right)$ are built on the same ring $\mathcal{C} = A/I$ of generalized constants. Let us suppose that the data of *problem* (P_G) *verify the conditions* $\varphi \in \mathcal{A}(\mathbb{R})$, $\psi \in \mathcal{A}(\mathbb{R})$, $f \in \mathrm{C}^{\infty}(\mathbb{R})$.

Then problem (P_G) *has a unique solution* u *in* $\mathcal{A}\left(\mathbb{R}^2\right)$.

Proof.

Let $u_\varepsilon = SP_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ the solution to $P_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ with the initial conditions $\varphi_\varepsilon \in \mathrm{C}^{\infty}(\mathbb{R})$ and $\psi_\varepsilon \in \mathrm{C}^{\infty}(\mathbb{R})$.

According to the previous result, it is enough to verify that $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$ for $u = [u_\varepsilon]$ to be solution to (P_G).

Any other solution v to (P_G) is in the form: $v = [v_\varepsilon]$, where $(v_\varepsilon)_\varepsilon$ verifies:

$$\begin{cases} \left(\dfrac{\partial^2(v_\varepsilon)}{\partial x \partial y} - F(.,.,v_\varepsilon)\right)_\varepsilon = (i_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right) \\ (v_\varepsilon(.,f(.)) - \varphi_\varepsilon(.))_\varepsilon = (\alpha_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}) \\ (\dfrac{\partial v_\varepsilon}{\partial y}(.,f(.)) - \psi_\varepsilon(.))_\varepsilon = (\beta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}) \end{cases}$$

and so the uniqueness of the solution to (P_G) will be the consequence of:

$$(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2).$$

a) Let us show that: $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$.

We will prove that:

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, (P_{K,l}(u_\varepsilon))_\varepsilon \in A_+.$$

let us proceed by induction showing first that we have:

a1)

$$\forall K \Subset \mathbb{R}^2, \left(P_{K,(0,0)}(u_\varepsilon)\right)_\varepsilon \in A_+,$$

with:

$$P_{K,(0,0)}(u_\varepsilon) = \sup_K |u_\varepsilon(x)| = \|u_\varepsilon\|_{\infty,K}$$

that is, the 0-ordered overestimation is verified. Let us put:

$$u_{0,\varepsilon}(x,y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x)$$

where χ_ε indicates a primitive of $\psi_\varepsilon \circ f^{-1}$.

According to the proposition 1.2.1.2., $\forall K \Subset \mathbb{R}^2, \exists K_\lambda \Subset \mathbb{R}^2, K \subset K_\lambda,$

$$\|u_\varepsilon\|_{\infty,K} \leq \|u_\varepsilon\|_{\infty,K_\lambda} \leq \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + \frac{\Phi_{\lambda,\varepsilon}}{m_\lambda} \exp[2\lambda m_\lambda \left(f(\lambda) - f(-\lambda)\right)].$$

We have $\left(\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\right)_\varepsilon \in A$ because $[\varphi_\varepsilon]$ and $[\psi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$.

$$m_\lambda = \sup_{(x,y)\in K_\lambda;\ t\in\mathbb{R}} \left|\frac{\partial F}{\partial z}(x,y,t)\right|$$

is a constant which depends entirely on F, K_λ.

$c(K_\lambda) = \dfrac{1}{m_\lambda} \exp[2\lambda m_\lambda \left(f(\lambda) - f(-\lambda)\right)]$ is a constant which depends entirely on F, f, K_λ.

$\Phi_{\lambda,\varepsilon} = \|F(.,.,0)\|_{\infty,K_\lambda} + m_\lambda \|u_{0,\varepsilon}\|_{\infty,K_\lambda}$ so:

$$\begin{aligned} &\frac{\Phi_{\lambda,\varepsilon}}{m_\lambda} \exp[2\lambda m_\lambda \left(f(\lambda) - f(-\lambda)\right)] \\ &\qquad\qquad = c(K_\lambda)\Phi_{\lambda,\varepsilon} \\ &\qquad\qquad = c(K_\lambda) \|F(.,.,0)\|_{\infty,K_\lambda} + \exp[2\lambda m_\lambda \left(f(\lambda) - f(-\lambda)\right)] \|u_{0,\varepsilon}\|_{\infty,K_\lambda}. \end{aligned}$$

$c_1(K_\lambda) = c(K_\lambda) \|F(.,.,0)\|_{\infty,K_\lambda}$ is a constant which depends entirely on F, K_λ;

$\exp[2\lambda m_\lambda \left(f(\lambda) - f(-\lambda)\right)]$ is a constant $c_2(K_\lambda)$ which depends entirely on K_λ, F, f.

Consequently:

$$\|u_\varepsilon\|_{\infty,K} \leq \|u_\varepsilon\|_{\infty,K_\lambda} \leq \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda) + c_2(K_\lambda) \|u_{0,\varepsilon}\|_{\infty,K_\lambda};$$

so:

$$\|u_\varepsilon\|_{\infty,K} \leq \|u_\varepsilon\|_{\infty,K_\lambda} \leq (1 + c_2(K_\lambda)) \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda).$$

$\left(\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\right)_\varepsilon \in A$ so $\left((1 + c_2(K_\lambda)) \|u_{0,\varepsilon}\|_{\infty,K_\lambda}\right)_\varepsilon \in A$ (if $(r_\varepsilon)_\varepsilon \in A$ then $(cr_\varepsilon)_\varepsilon \in A$) and as $c_1(K_\lambda)$ is a constant ($(1)_\varepsilon \in A$), we deduce that:

$$((1 + c_2(K_\lambda)) \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda))_\varepsilon \in A.$$

A being stable by overestimation: $\left(\|u_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in A_+$ and so: $\left(\|u_\varepsilon\|_{\infty,K}\right)_\varepsilon \in A_+$ that is: $\left(P_{K,0}\left(u_\varepsilon\right)\right)_\varepsilon \in A_+$.

a2) Let us show that:

$$\left(P_{K,1}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

We have:

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int_{f(x)}^{y} F(x,\eta,u_\varepsilon(x,\eta))d\eta,$$

hence:

$$\begin{aligned} P_{K,(1,0)}(u_\varepsilon) = \left\|\frac{\partial u_\varepsilon}{\partial x}\right\|_{\infty,K} &= \sup_K \left|\frac{\partial u_\varepsilon}{\partial x}(x,y)\right| \\ &\leq \sup_K \left|\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y)\right| + (f(\lambda)-f(-\lambda))\left(\sup_{K_\lambda}|F(x,\eta,u_\varepsilon(x,\eta))|\right). \end{aligned}$$

$\mathcal{A}\left(\mathbb{R}^2\right)$ being stable under F, there exist $C>0$ such that:

$$P_{K_\lambda,(0,0)}(F(.,.,u_\varepsilon)) \leq P_{K_\lambda,0}(F(.,.,u_\varepsilon)) \leq C. \quad (1)$$

We have:

$$\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) = f'(x)\psi_\varepsilon(x) + \varphi'_\varepsilon(x),$$

hence:

$$\left(\left\|\frac{\partial u_{0,\varepsilon}}{\partial x}\right\|_{\infty,K}\right)_\varepsilon \in A_+$$

because $[\varphi_\varepsilon]$ and $[\psi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$.

So:

$$P_{K,(1,0)}(u_\varepsilon) \leq \left\|\frac{\partial u_{0,\varepsilon}}{\partial x}\right\|_{\infty,K} + C\left(f(\lambda)-f(-\lambda)\right).$$

A being stable by overestimation: $\left(P_{K,(1,0)}(u_\varepsilon)\right)_\varepsilon \in A^+$.

We have:

$$\frac{\partial u_\varepsilon}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) - \int_{x}^{f^{-1}(y)} F(\xi,y,u_\varepsilon(\xi,y))d\xi,$$

so:

$$P_{K,(0,1)}(u_\varepsilon) = \left\| \frac{\partial u_\varepsilon}{\partial y} \right\|_{\infty,K} = \sup_K \left| \frac{\partial u_\varepsilon}{\partial y}(x,y) \right| \leq \sup_K \left| \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) \right| + 2\lambda \left(\sup_{K_\lambda} |F(x,\eta,u_\varepsilon(x,\eta))| \right).$$

We have:

$$\frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) = \psi_\varepsilon(f^{-1}(y),$$

so:

$$\left(\left\| \frac{\partial u_{0,\varepsilon}}{\partial y} \right\|_{\infty,K} \right)_\varepsilon \in A_+$$

because $[\psi_\varepsilon]$ is element of $\mathcal{A}(\mathbb{R})$; hence:

$$P_{K,(0,1)}(u_\varepsilon) \leq \left\| \frac{\partial u_{0,\varepsilon}}{\partial y} \right\|_{\infty,K} + C2\lambda$$

and so, like previously:

$$\left(\left\| \frac{\partial u_\varepsilon}{\partial y} \right\|_{\infty,K} \right)_\varepsilon \in A_+.$$

a3) Induction.

Let us suppose that, for every $l \leq n$, we have: $(P_{K,l}(u_\varepsilon))_\varepsilon \in A_+$ and let us show that involves $(P_{K,n+1}(u_\varepsilon))_\varepsilon \in A_+$.

In fact we have:

$$P_{K,n+1} = \max(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n})$$

with:

$$P_{1,n} = P_{K,(n+1,0)},$$

$$P_{2,n} = P_{K,(0,n+1)},$$

$$P_{3,n} = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)},$$

$$P_{4,n} = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}.$$

a3.1) First let us show that for every $n \in \mathbb{N}$,

$$\left(P_{1,n}(u_\varepsilon)\right)_\varepsilon \in A_+,\ \left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

As we have:

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int_{f(x)}^{y} F(x,\eta,u_\varepsilon(x,\eta))d\eta,$$

we deduce that:

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}(x,y) = \frac{\partial^2 u_{0,\varepsilon}}{\partial x^2}(x,y) - f'(x)F(x,f(x),\varphi(x)) + \int_{f(x)}^{y} \frac{\partial}{\partial x}F(x,\eta,u_\varepsilon(x,\eta))d\eta$$

and by successive derivations, for $n \geq 1$:

$$\begin{aligned}\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) &= \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) \\ &\quad - \sum\nolimits_{j=0}^{n-1} C_n^j f^{(n-j)}(x)\frac{\partial^j}{\partial x^j}F(x,f(x),\varphi_\varepsilon(x)) + \int_{f(x)}^{y} \frac{\partial^n}{\partial x^n}F(x,\eta,u_\varepsilon(x,\eta))d\eta.\end{aligned}$$

As we have taken $K \subset K_\lambda$, we can write:

$$\begin{aligned}\sup_{(x,y)\in K}\left|\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y)\right| &\leq \\ &\left\|\frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}\right\|_{\infty,K} + \sup_{x\in[-\lambda,\lambda]}\sum\nolimits_{j=0}^{n-1} C_n^j \left|f^{(n-j)}(x)\right| \left|\frac{\partial^j}{\partial x^j}F(x,f(x),\varphi_\varepsilon(x))\right| \\ &\quad + \left(f\left(\lambda\right) - f\left(-\lambda\right)\right)\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}F(x,y,u_\varepsilon(x,y))\right|\right).\end{aligned}$$

We have:

$$\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}F(x,y,u_\varepsilon(x,y))\right|\right) = P_{K,(n,0)}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)),$$

and:

$$\begin{aligned}\sup_{x\in[-\lambda,\lambda]}\left|\frac{\partial^j}{\partial x^j}F(x,f(x),\varphi_\varepsilon(x))\right| &\leq P_{K,(j,0)}(F(.,.,u_\varepsilon)) \\ &\leq P_{K,(n,0)}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)),\end{aligned}$$

moreover:

$$\left(\left\|\frac{\partial^{n+1}u_{0,\varepsilon}}{\partial x^{n+1}}\right\|_{\infty,K}\right)_{\varepsilon}\in A_{+},$$

because $[\varphi_\varepsilon]$ and $[\psi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$.

According to the hypothesis of stability, a simple calculation shows then that, for every $K \Subset \mathbb{R}^2$,

$$\left(P_{K,(n+1,0)}\left(u_\varepsilon\right)\right)_\varepsilon \in A_+.$$

Let us show that, for every $n \in \mathbb{N}$, $\left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A_+$.

As we have:

$$\frac{\partial u_\varepsilon}{\partial y}(x,y)=\frac{\partial u_{0,\varepsilon}}{\partial y}(x,y)-\int_x^{f^{-1}(y)}F(\xi,y,u_\varepsilon(\xi,y))d\xi,$$

we deduce that:

$$\begin{aligned}\frac{\partial^2 u_\varepsilon}{\partial y^2}(x,y)=\frac{\partial^2 u_{0,\varepsilon}}{\partial y^2}(x,y)&-\left(\left(f^{-1}\right)'(y)\right)F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y)))\\&-\int_x^{f^{-1}(y)}\frac{\partial}{\partial y}F(\xi,y,u_\varepsilon(\xi,y))d\xi\end{aligned}$$

and by successive derivations, for $n \geq 1$:

$$\begin{aligned}\frac{\partial^{n+1}u_\varepsilon}{\partial y^{n+1}}(x,y)=\frac{\partial^{n+1}u_{0,\varepsilon}}{\partial y^{n+1}}(x,y)&-\int_x^{f^{-1}(y)}\frac{\partial^n}{\partial y^n}F(\xi,y,u_\varepsilon(\xi,y))d\xi\\&-\sum_{j=0}^{n-1}C_n^j\left(f^{-1}\right)^{(n-j)}(y)\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y))).\end{aligned}$$

As we have taken $K \subset K_\lambda$, we can write:

$$\begin{aligned}&\sup_{(x,y)\in K}\left|\frac{\partial^{n+1}u_\varepsilon}{\partial y^{n+1}}(x,y)\right|\\&\qquad\leq\left\|\frac{\partial^{n+1}u_{0,\varepsilon}}{\partial y^{n+1}}\right\|_{\infty,K}+(2\lambda)\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|\right)\\&\qquad+\sup_{y\in[f(-\lambda),f(\lambda)]}\sum_{j=0}^{n-1}C_n^j\left|\left(f^{-1}\right)^{(n-j)}(y)\right|\left|\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y)))\right|.\end{aligned}$$

We have:

$$\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|\right)=P_{K,(0,n)}(F(.,.,u_\varepsilon))$$

$$\leq P_{K,n}(F(.,.,u_\varepsilon))$$

and:

$$\sup_{y\in[f(-\lambda),f(\lambda)]}\left|\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y)))\right|\leq\left(\sup_{(x,y)\in K}\left|\frac{\partial^i}{\partial y^i}F(x,y,u_\varepsilon(x,y))\right|\right)$$

$$\leq P_{K,i}(F(.,.,u_\varepsilon))\leq P_{K,n}(F(.,.,u_\varepsilon)).$$

According to the hypothesis of stability, a simple calculation shows then that, for every $K\Subset\mathbb{R}^2$ and for every $n\in\mathbb{N}$,

$$\left(P_{K,(0,n+1)}\left(u_\varepsilon\right)\right)_\varepsilon\in A_+.$$

a3.2) For $\alpha+\beta=n$ and $\beta\geq 1$, we have now:

$$P_{K,(\alpha+1,\beta)}(u_\varepsilon)=\sup_{(x,y)\in K}\left|D^{(\alpha+1,\beta)}u_\varepsilon\left(x,y\right)\right|=\sup_{(x,y)\in K}\left|D^{(\alpha,\beta-1)}D^{(1,1)}u_\varepsilon\left(x,y\right)\right|$$

$$=\sup_{(x,y)\in K}\left|D^{(\alpha,\beta-1)}F(x,y,u_\varepsilon\left(x,y\right))\right|=P_{K,(\alpha,\beta-1)}(F(.,.,u_\varepsilon))$$

$$\leq P_{K,n-1}(F(.,.,u_\varepsilon))\leq P_{K,n}(F(.,.,u_\varepsilon)).$$

So we finally have:

$$P_{3,n}(u_\varepsilon)=\sup_{\alpha+\beta=n;\beta\geq 1}P_{K,(\alpha+1,\beta)}(u_\varepsilon)\leq P_{K,n}(F(.,.,u_\varepsilon))$$

and the hypothesis of stability then assures that:

$$\left(P_{3,n}(u_\varepsilon)\right)_\varepsilon\in A_+.$$

In the same way, for $\alpha+\beta=n$ and $\alpha\geq 1$, we have:

$$\begin{aligned} P_{K,(\alpha,\beta+1)}(u_\varepsilon) &= \sup_{(x,y)\in K}\left|D^{(\alpha,\beta+1)}u_\varepsilon\left(x,y\right)\right| = \sup_{(x,y)\in K}\left|D^{(\alpha-1,\beta)}D^{(1,1)}u_\varepsilon\left(x,y\right)\right| \\ &= \sup_{(x,y)\in K}\left|D^{(\alpha-1,\beta)}F(x,y,u_\varepsilon\left(x,y\right))\right| = P_{K,(\alpha-1,\beta)}(F(.,.,u_\varepsilon)) \\ &\leq P_{K,n-1}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)). \end{aligned}$$

So we have:

$$P_{4,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\varepsilon) \leq P_{K,n}(F(.,.,u_\varepsilon))$$

and the hypothesis of stability then assures that:

$$\left(P_{4,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

Finally, we clearly have:

$$\left(P_{K,n+1}(u_\varepsilon)\right)_\epsilon \in A_+.$$

b) Let us show that u is the unique solution to (P_G).

Let $v=[v_\varepsilon]$ an other solution to (P_G).

There are $(i_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$, $(\alpha_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}\right)$, $(\beta_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}\right)$, such that:

$$\begin{cases} \dfrac{\partial^2 v_\varepsilon}{\partial x\partial y}(x,y) = F(x,y,v_\varepsilon(x,y)) + i_\varepsilon(x,y) \\ v_\varepsilon\left(x,f(x)\right) = \varphi_\varepsilon(x) + \alpha_\varepsilon(x) \\ \dfrac{\partial v_\varepsilon}{\partial y}\left(x,f(x)\right) = \psi_\varepsilon(x) + \beta_\varepsilon(x). \end{cases}$$

It is easy to see that:

$$\left(\iint_{D(x,y,f)} i_\varepsilon(\xi,\eta))d\xi d\eta\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right).$$

So there is $(j_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$ such that:

$$v_\varepsilon(x,y) = v_{0,\varepsilon}(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta + j_\varepsilon(x,y),$$

with: $v_{0,\varepsilon}(x,y) = u_{0,\varepsilon}(x,y) + \theta_\varepsilon(x,y)$; where: $u_{0,\varepsilon}(x,y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x)$ and:

$$\theta_\varepsilon(x,y) = B_\varepsilon(y) - B_\varepsilon(f(x)) + \alpha_\varepsilon(x)$$

where B_ε is a primitive of $\beta_\varepsilon \circ f^{-1}$. So $(\theta_\varepsilon)_\varepsilon$ belongs to $\mathcal{N}(\mathbb{R}^2)$.

So there is $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ such that:

$$v_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \sigma_\varepsilon(x,y) - \iint_{D(x,y,f)} F(\alpha,\beta,v_\varepsilon(\alpha,\beta))d\alpha d\beta.$$

b1) Let us put $w_\varepsilon = v_\varepsilon - u_\varepsilon$ and let us show that: $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$.

We have to prove that:

$$\forall K \Subset \mathbb{R}^2, \forall n \in \mathbb{N}, (P_{K,n}(w_\varepsilon))_\varepsilon \in I_A^+.$$

Let us proceed by induction showing first that we have:

$$(P_{K,1}(w_\varepsilon))_\varepsilon \in I_A.$$

We have:

$$w_\varepsilon(x,y) = \iint_{D(x,y,f)} \left(-F(\xi,\eta,v_\varepsilon(\xi,\eta)) + F(\xi,\eta,u_\varepsilon(\xi,\eta))\right) d\xi d\eta + \sigma_\varepsilon(x,y),$$

however:

$$\begin{aligned} &F(\xi,\eta,v_\varepsilon(\xi,\eta)) - F(\xi,\eta,u_\varepsilon(\xi,\eta)) = \\ &\qquad (v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta)) \left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta)))d\theta\right), \end{aligned}$$

so:

$$w_\varepsilon(x,y) = -\iint_{D(x,y,f)} w_\varepsilon(\xi,\eta) \left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(w_\varepsilon(\xi,\eta)))d\theta\right) d\xi d\eta + \sigma_\varepsilon(x,y). \qquad (3)$$

Let $(x,y) \in K_\lambda$, since $D(x,y,f) \subset K_\lambda$, if $y \geq f(x)$, we have:

$$\begin{aligned} |w_\varepsilon(x,y)| &\leq m_\lambda \int_x^{f^{-1}(y)} \int_{f(\xi)}^y |w_\varepsilon(\xi,\eta)|\, d\xi d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda} \\ &\leq m_\lambda \int_{-\lambda}^{+\lambda} \int_{f(x)}^y |w_\varepsilon(\xi,\eta)|\, d\xi d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda} . \end{aligned}$$

Let us put: $e_\varepsilon(y) = \sup\limits_{\xi \in [-\lambda;+\lambda]} |w_\varepsilon(\xi,y)|$, then:

$$|w_\varepsilon(x,y)| \leq m_\lambda 2\lambda \int_{f(-\lambda)}^y e_\varepsilon(\eta) d\eta + \|\sigma_\varepsilon\|_{\infty,k_\lambda} ,$$

we deduce that:

$$\forall y \in [f(-\lambda); f(+\lambda)] \text{, si } y \geq f(x),\ e_\varepsilon(y) \leq m_\lambda 2\lambda \int_{f(-\lambda)}^y e_\varepsilon(\eta) d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda} .$$

Reminder: *Gronwall Lemma.*

Let α*:* $[t_0,t_1] \to \mathbb{R}$ *a continuous positive function and* u_0 *a positive constant.*

Every function f *such that:* $0 \leq f(t) \leq u_0 + \int_{t_0}^t f(s)\alpha(s) ds$*, verifies the inequalities:*

$$0 \leq f(t) \leq u_0 \exp\left(\int_{t_0}^t \alpha(s) ds\right).$$

Thus, according to Gronwall lemma:

$$\forall y \in [f(-\lambda); f(+\lambda)] \text{, if } y \geq f(x),\ e_\varepsilon(y) \leq \left(\exp(\int_{f(-\lambda)}^y m_\lambda 2\lambda d\eta)\right) \|\sigma_\varepsilon\|_{\infty,K_\lambda} .$$

We obtain the same result for $y \leq f(x)$, hence, for every $y \in [f(-\lambda); f(+\lambda)]$:

$$\begin{aligned} e_\varepsilon(y) &\leq (\exp(m_\lambda 2\lambda\, (y - f(-\lambda)))\, \|\sigma_\varepsilon\|_{\infty,K_\lambda} \\ &\leq [\exp(m_\lambda (2\lambda)(f(\lambda) - f(-\lambda))]\, \|\sigma_\varepsilon\|_{\infty,K_\lambda} , \end{aligned}$$

consequently:

$$\|w_\epsilon\|_{\infty,K_\lambda} \leq [\exp(m_\lambda (2\lambda)(f(\lambda) - f(-\lambda))]\, \|\sigma_\varepsilon\|_{\infty,K_\lambda} ,$$

$(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ so $\left(\|\sigma_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in I_A$.

$[\exp(m_\lambda(2\lambda)(f(\lambda)-f(-\lambda))]$ is a constant, consequently $\left(\|w_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in I_A$.

Which involves the 0-ordered estimation.

b2) Induction.

Let us suppose that, for every $l \leq n$, we have: $(P_{K,l}(w_\varepsilon))_\varepsilon \in I_A^+$ and let us show that involves $(P_{K,n+1}(w_\varepsilon))_\varepsilon \in I_A^+$.

b2.1) First let us show that for every $n \in \mathbb{N}$:

$$(P_{1,n}(w_\varepsilon))_\varepsilon \in I_A^+.$$

We have:

$$\frac{\partial w_\varepsilon}{\partial x}(x,y) = \frac{\partial \sigma_\varepsilon}{\partial x}(x,y) + \int_{f(x)}^{y} \left(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta))\right) d\eta$$

and by successive derivations, for $n \geq 1$:

$$\begin{aligned}\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) &= \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) \\ &\quad - \sum\nolimits_{j=0}^{n-1} C_n^j f^{(n-j)}(x) \frac{\partial^j}{\partial x^j} F(x,f(x),\varphi_\varepsilon(x)) + \int_{f(x)}^{y} \frac{\partial^n}{\partial x^n} F(x,\eta,u_\varepsilon(x,\eta)) d\eta.\end{aligned}$$

So:

$$\begin{aligned}\frac{\partial^{n+1} w_\varepsilon}{\partial x^{n+1}}(x,y) &= \frac{\partial^{n+1} \sigma_\varepsilon}{\partial x^{n+1}}(x,y) + \delta_\varepsilon(x) + \\ &\qquad \int_{f(x)}^{y} \frac{\partial^n}{\partial x^n}\left(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta))\right) d\eta,\end{aligned}$$

with:

$$\delta_\varepsilon(x) = \sum\nolimits_{j=0}^{n-1} C_n^j f^{(n-j)}(x) \left(\frac{\partial^j}{\partial x^j} F(x,f(x),\varphi_\varepsilon(x)) - \frac{\partial^j}{\partial x^j} F(x,f(x),\varphi_\varepsilon(x)+\alpha_\varepsilon(x))\right),$$

$(\delta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$. Hence:

$$\begin{aligned}P_{K,(n+1,0)}(w_\varepsilon) &\leq P_{K,(n+1,0)}(\sigma_\varepsilon) + \sup_{x\in[-\lambda,\lambda]} |\delta_\varepsilon(x)| \\ &\quad + (f(\lambda) - f(-\lambda)) \left(\sup_{(x,y)\in K} \left|\frac{\partial^n}{\partial x^n}\left(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta))\right)\right|\right).\end{aligned}$$

We have:

$$\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}\left(F(x,\eta,v_\varepsilon(x,\eta))-F(x,\eta,u_\varepsilon(x,\eta))\right)\right|\right)=P_{K,(n,0)}\left(F(.,.,v_\varepsilon)-F(.,.,u_\varepsilon)\right)$$

$$\leq P_{K,n}\left(F(.,.,v_\varepsilon)-F(.,.,u_\varepsilon)\right).$$

According to the hypothesis of stability, for every $K\Subset\mathbb{R}^2$:

$$\left(P_{K,(n+1,0)}(w_\varepsilon)\right)_\epsilon\in I_A^+.$$

Let us show that, for every $n\in\mathbb{N}$, $\left(P_{2,n}(w_\varepsilon)\right)_\varepsilon\in I_A^+$.

We have:

$$\frac{\partial^{n+1}u_\varepsilon}{\partial y^{n+1}}(x,y)=\frac{\partial^{n+1}u_{0,\varepsilon}}{\partial y^{n+1}}(x,y)-\sum\nolimits_{j=0}^{n-1}C_n^j\left(f^{-1}\right)^{(n-j)}(y)\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y)))$$
$$-\int_x^{f^{-1}(y)}\frac{\partial^n}{\partial y^n}F(\xi,y,u_\varepsilon(\xi,y))d\xi.$$

So:

$$\frac{\partial^{n+1}w_\varepsilon}{\partial y^{n+1}}(x,y)=\frac{\partial^{n+1}\sigma_\varepsilon}{\partial y^{n+1}}(x,y)+\mu_\varepsilon(y)-\int_x^{f^{-1}(y)}\left(\frac{\partial^n}{\partial y^n}F(x,y,v_\varepsilon(x,y))-\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right)d\xi,$$

with:

$$\mu_\varepsilon(y)=\sum\nolimits_{j=0}^{n-1}C_n^j\left(f^{-1}\right)^{(n-j)}(y)\left(\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y)))\right)$$
$$-\sum\nolimits_{j=0}^{n-1}C_n^j\left(f^{-1}\right)^{(n-j)}(y)\left(\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y))+\alpha_\varepsilon(f^{-1}(y))\right).$$

$(\mu_\varepsilon)_\varepsilon\in\mathcal{N}(\mathbb{R})$. Hence:

$$P_{K,(0,n+1)}(w_\varepsilon)\leq P_{K,(0,n+1)}(\sigma_\varepsilon)+\sup_{y\in[f(-\lambda),f(\lambda)]}|\mu_\varepsilon(y)|$$
$$+(2\lambda)\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,v_\varepsilon(x,y))-\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|\right).$$

We have:

$$\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,v_\varepsilon(x,y))-\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|\right)=P_{K,(0,n)}(F(.,.,v_\varepsilon)-F(.,.,u_\varepsilon))$$

$$\leq P_{K,(0,n)}(F(.,.,v_\varepsilon)-F(.,.,u_\varepsilon)).$$

according to the hypothesis of stability, for every $K \Subset \mathbb{R}^2$:

$$\left(P_{K,(0,n+1)}(w_\varepsilon)\right)_\epsilon \in I_A.$$

b2 2)

For $\alpha+\beta=n$ and $\beta \geq 1$, we have:

$$P_{K,(\alpha+1,\beta)}(w_\varepsilon) = P_{K,(\alpha,\beta-1)}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)).$$

Finally we have:

$$P_{3,n}(w_\varepsilon) = \sup_{\alpha+\beta=n,\beta\geq 1} P_{K,(\alpha+1,\beta)}(w_\varepsilon) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon))$$

and the hypothesis of stability then assures that:

$$(P_{3,n}(w_\varepsilon))_\varepsilon \in I_A^+.$$

In the same way, for $\alpha+\beta=n$ et $\alpha \geq 1$, we have:

$$P_{K,(\alpha,\beta+1)}(w_\varepsilon) = P_{K,(\alpha-1,\beta)}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)).$$

So we finally have:

$$P_{4,n}(w_\varepsilon) = \sup_{\alpha+\beta=n,\alpha\geq 1} P_{K,(\alpha,\beta+1)}(w_\varepsilon) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon))$$

and the hypothesis of stability assures that:

$$(P_{4,n}(w_\varepsilon))_\varepsilon \in I_A^+.$$

So for every $l \leq n+1$, we have:

$$(P_{K,l}(w_\varepsilon))_\epsilon \in I_A^+.$$

Proceeding by induction we obtain, for every $n \in \mathbb{N}$:

$$\left(P_{K,n}(w_\varepsilon)\right)_\epsilon \in I_A^+.$$

So $(w_\epsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$; consequently u is the unique solution to (P_G). $\square$

Chapter 5

Qualitative study of the solution

5.1 Parametric singular spectrum of the solution to the Cauchy problem

5.1.1 Relation between the $\mathcal{D}'$-parametric singular spectrum of solution u and the $\mathcal{D}'$-parametric singular spectrum of u_0

5.1.1.1. Theorem

We put $u_0 = [u_{0,\varepsilon}]$ *with*

$$u_{0,\varepsilon}(x,y) = \chi_\varepsilon\,(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x)$$

where χ_ε indicates a primitive of $\psi_\varepsilon \circ f^{-1}$, and we *suppose that:*

(H_2) $\quad \forall K \Subset \mathbb{R}^2, \mathcal{M}_F(K) = \sup\limits_{(x,y)\in K, z\in\mathbb{R}} |F(x,y,z)| < +\infty.$

Then the restriction to the parametric singular support of u_0 *of the* $\mathcal{D}'$*-parametric singular spectrum of the solution* u *to the Cauchy problem* (P_G) *is included in the restriction to the parametric singular support of* u_0 *of the* $\mathcal{D}'$*-parametric singular spectrum of* u_0*. In other words, over the singular support of* u_0*, there is no increase in the distributional* *singularities of* u *in comparison with those of* u_0.

Proof.

Let $(x_0, y_0) = X \in S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0$ and $r \in N_{\mathcal{D}',X}(u_0)$. It results from the definitions that we have: $\Sigma_{\mathcal{D}',X}(u_0) \neq \emptyset$, and so that $N_{\mathcal{D}',X}(u_0) \subset]0, +\infty[$ which involve that $r > 0$.

Let us show that we then have: $r \in N_{\mathcal{D}',X}(u)$.

From the definition of $N_{\mathcal{D}',X}(u_0)$, there exists a neighbourhood V_X of X such that:

$$\lim_{\varepsilon \to 0} (\varepsilon^r u_{\varepsilon|_{V_X}}) \in \mathcal{D}'(V_X).$$

Let $g \in \mathcal{D}(V_X)$. So, there exists some distribution $T \in \mathcal{D}'(V_X)$ such that:

$$\lim_{\varepsilon \to 0} \iint_{V_X} \varepsilon^r u_{0,\varepsilon}(x, y) g(x, y) dx dy = T(g).$$

Let us show that:

$$\iint_{V_X} \varepsilon^r \left[u_\varepsilon(x, y) - u_{0,\varepsilon}(x, y)\right] g(x, y) dx dy$$

has 0 for limit when ε tends to 0.

Supposing moreover that $y \geq f(x)$.

As $u_\varepsilon(x, y) - u_{0,\varepsilon}(x, y) = -\iint_{D(x,y,f)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\xi d\eta$

and that (with the above notations):

$$\begin{aligned}
&\left| \iint_{V_X} \left[\iint_{D(x,y,f)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\xi d\eta \right] g(x, y) dx dy \right| \\
&\qquad \leq \mathcal{M}_F(suppg) \left| \iint_{suppg} \left[\iint_{D(x,y,f)} d\xi d\eta \right] g(x, y) dx dy \right| \\
&\qquad \leq \mathcal{M}_F(suppg) \left| \iint_{suppg} \left(A(x, y)\right) g(x, y) dx dy \right| \\
&\qquad \leq \mathcal{M}_F(suppg) \left| \iint_{suppg} \left(2\lambda \left|y\right|\right) g(x, y) dx dy \right| \\
&\qquad \leq 2\lambda \mathcal{M}_F(suppg) \iint_{suppg} \left|y\right| \left|g(x, y)\right| dx dy < +\infty,
\end{aligned}$$

then, we have:

$$\begin{aligned}
\limsup_{\varepsilon\to 0}&\left|\iint_{V_X}\varepsilon^r\left[u_\varepsilon(x,y)-u_{0,\varepsilon}(x,y)\right]g(x,y)dxdy\right| \\
&\leq \limsup_{\varepsilon\to 0}\varepsilon^r\left|\iint_{V_X}\left[\iint_{D(x,y,f)}F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta\right]g(x,y)dxdy\right| \\
&\leq \limsup_{\varepsilon\to 0}\varepsilon^r\left[2\lambda\left(\mathcal{M}_F(suppg)\right)\iint_{suppg}|y|\,|g(x,y)|\,dxdy\right]=0,
\end{aligned}$$

because $r \neq 0$. And so:

$$\lim_{\varepsilon\to 0}\iint_{V_X}\varepsilon^r u_\varepsilon(x,y)g(x,y)dxdy=\lim_{\varepsilon\to 0}\iint_{V_X}\varepsilon^r u_{0,\varepsilon}(x,y)g(x,y)dxdy=T(g).$$

It follows that:

$$\lim_{\varepsilon\to 0}(\varepsilon^r u_{\varepsilon|_{V_X}})=\lim_{\varepsilon\to 0}(\varepsilon^r u_{0,\varepsilon|_{V_X}})\in\mathcal{D}'(V_X).$$

So we have: $r \in N_{\mathcal{D}',X}(u)$, which proves the inclusion: $N_{\mathcal{D}',X}(u_0)\subset N_{\mathcal{D}',X}(u)$, and consequently, $\Sigma_{\mathcal{D}',X}(u)\subset\Sigma_{\mathcal{D}',X}(u_0)$, then we have:

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u\;{}_{S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u_0}\subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u\;{}_{0\;S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u_0}.$$

□

5.1.2 Examples.

Let us take $g\in\mathcal{D}(\mathbb{R})$, $g\geq 0$, $\int_{\mathbb{R}}g(x)dx=1$.

With the above notations we have: $\varphi=[\varphi_\varepsilon]$ and $\Psi=[\Psi_\varepsilon]$ where χ_ε is a primitive of $\psi_\varepsilon\circ f^{-1}$ and:

$$u_{0,\varepsilon}(x,y)=\chi_\varepsilon(y)-\chi_\varepsilon(f(x))+\varphi_\varepsilon(x).$$

$f(x)=ax$, $a>0$. Let us consider the following cases:

1)

$$\chi_\varepsilon(y)=\varepsilon^{-1}g(y\varepsilon^{-1})\text{ and }\varphi_\varepsilon(x)=\varepsilon^{-1}g(x\varepsilon^{-1}),$$

so:

$$\chi_\varepsilon(f(x)) = \varepsilon^{-1} g(f(x)\varepsilon^{-1}) = \varepsilon^{-1}\varphi(ax\varepsilon^{-1}).$$

$N_{\mathcal{D}',X}(u_0) = [1, +\infty[$, then we have: $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0, 1[$.

2)

$\chi_\varepsilon(x) = \varepsilon^{-1} g(x\varepsilon^{-1})$ and $\varphi_\varepsilon(x) = \varepsilon^{-2}\varphi(x\varepsilon^{-1}) = \varepsilon^{-1}\left[\varepsilon^{-1}\varphi(x\varepsilon^{-1})\right]$.

$N_{\mathcal{D}',X}(u_0) = [2, +\infty[$, then we have: $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0, 2[$.

3)

$$\chi_\varepsilon(x) = g(x\varepsilon^{-1}) \text{ and } \varphi_\varepsilon(x) = g(x\varepsilon^{-1}) = \varepsilon[\varepsilon^{-1} g(x\varepsilon^{-1})]$$

$N_{\mathcal{D}',X}(u_0) = [0, +\infty[$, then we have: $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times \varnothing$. □

5.2 Qualitative study of the solution. Case: $F = 0$

5.2.1 Terms of the problem

We search for a generalized solution u to the following Cauchy problem:

$$(P_G)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = 0 \\ u_{|\gamma} = \varphi \\ \dfrac{\partial u}{\partial y}_{|\gamma} = \psi \end{cases}$$

considering as data the curve γ of equation $y = f(x)$.

Let:

$$P_\infty(\varphi_\varepsilon, \psi_\varepsilon)\begin{cases} \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y}(x, y) = 0 \\ u_\varepsilon(x, f(x)) = \varphi_\varepsilon(x) \\ \dfrac{\partial u_\varepsilon}{\partial y}(x, f(x)) = \psi_\varepsilon(x). \end{cases}$$

With the above notations, we have:

$$\varphi = [\varphi_\varepsilon], \ \psi = [\chi_\varepsilon] \text{ and } u_\varepsilon(x, y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x).$$

5.2.2 Qualitative study of the solution. Case: $F = 0$, $f(x) = ax$, $(a > 0)$

Case: $f(x) = ax$**,** $(a > 0)$**,** $\varphi \sim \delta$**,** $\psi \sim \delta$**,**

the association is defined in the following way:

Considering g in the space $\mathcal{D}(\mathbb{R})$, a even function, verifying $\int_{\mathbb{R}} g(\xi)d\xi = 1$. Let us put: $\varphi_\varepsilon(x) = \dfrac{1}{\varepsilon}g(\dfrac{x}{\varepsilon}) = \psi_\varepsilon(x)$. Then $(\varphi_\varepsilon)_\varepsilon$ and $(\psi_\varepsilon)_\varepsilon$ have, in a distributional sense, δ as limit. So $\varphi = [\varphi_\varepsilon]$ and $\psi = [\psi_\varepsilon]$ are actually associated to δ.

The solution to $P_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ is defined by: $u_\varepsilon(x,y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x)$, with:

$$\chi_\varepsilon(y) = \int_0^y \psi_\varepsilon\left(f^{-1}(\eta)\right) d\eta = \int_0^y \psi_\varepsilon\left(\frac{\eta}{a}\right) d\eta = a\int_0^{\frac{y}{a}} \psi_\varepsilon(t)\, dt = a\left(\Psi_\varepsilon\left(\frac{y}{a}\right) - \Psi_\varepsilon(0)\right)$$

where Ψ_ε is a primitive of ψ_ε.

So:

$$u_\varepsilon(x,y) = a\Psi_\varepsilon\left(\frac{y}{a}\right) - a\Psi_\varepsilon(x) + \varphi_\varepsilon(x).$$

We can choose Ψ_ε such that $\Psi_\varepsilon(0) = \dfrac{1}{2}$, in such a way that:

$$\lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} \Psi_\varepsilon = Y \text{ and } \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} \left(y \mapsto \Psi_\varepsilon\left(\frac{y}{a}\right)\right) = Y.$$

Then we have:

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}],$$

with:

$$\begin{cases} [w_{\varepsilon,1}] \sim a(1_x \otimes Y_y) \\ [w_{\varepsilon,2}] \sim -a(Y_x \otimes 1_y) \\ [w_{\varepsilon,3}] \sim \delta_x \otimes 1_y. \end{cases}$$

□

Case: $f(x) = ax$**,** $(a > 0)$**,** $\varphi \sim \delta$**,** $\psi = \Psi'$**, with** $\Psi \sim \delta$**,**

the association being achieved in such a way that :

Considering g in the space $\mathcal{D}(\mathbb{R})$, verifying $\int_{\mathbb{R}} g(\xi)d\xi = 1$. Let us put : $\varphi_\varepsilon(x) = \dfrac{1}{\varepsilon}g(\dfrac{x}{\varepsilon}) = \Psi_\varepsilon(x)$. Then $(\varphi_\varepsilon)_\varepsilon$ and $(\Psi_\varepsilon)_\varepsilon$ converge in a distributional sense to δ. Then we put $\varphi = [\varphi_\varepsilon]$ and $\Psi = [\Psi_\varepsilon]$.

The solution to $P_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ is defined by:

$$u_\varepsilon(x,y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x) = a\Psi_\varepsilon\left(\frac{y}{a}\right) - a\Psi_\varepsilon(x) + \varphi_\varepsilon(x).$$

We calculate:

$$\frac{1}{a}\int \Psi_\varepsilon\left(\frac{y}{a}\right)dy = \frac{1}{a}\frac{1}{\varepsilon}\int g(\frac{y}{a\varepsilon})dy = \frac{1}{\varepsilon}\int g(\frac{x}{\varepsilon})dx = 1.$$

It follows that:

$$\frac{1}{a}\lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}}\left(y \mapsto \Psi_\varepsilon\left(\frac{y}{a}\right)\right) = \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}}\Psi_\varepsilon = \delta.$$

So:

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}],$$

with:

$$\begin{cases} [w_{\varepsilon,1}] \sim a^2(1_x \otimes \delta_y) \\ [w_{\varepsilon,2}] \sim -a(\delta_x \otimes 1_y) \\ [w_{\varepsilon,3}] \sim \delta_x \otimes 1_y, \end{cases}$$

hence:

$$u \sim a^2(1_x \otimes \delta_y) - a(\delta_x \otimes 1_y) + \delta_x \otimes 1_y.$$

□

Case: $f(x) = ax$**,** $(a > 0)$**,** $\varphi \sim S$**,** $\psi = \Psi'$ **and** $\Psi \sim T$**;** $S \in \mathcal{D}'(\mathbb{R})$**,** $T \in \mathcal{D}'(\mathbb{R})$**,**

choosing:

$$\varphi = [g_\varepsilon * S] \text{ and } \Psi = [g_\varepsilon * T]$$

the association being achieved, since:

$$\lim_{\substack{\varepsilon \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} 0}} (g_\varepsilon * S)_\varepsilon = S \text{ and } \lim_{\substack{\varepsilon \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} 0}} (g_\varepsilon * T)_\varepsilon = T.$$

So we have here:

$$\begin{aligned} u_\varepsilon(x,y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x) &= a\Psi_\varepsilon\left(\frac{y}{a}\right) - a\Psi_\varepsilon(x) + \varphi_\varepsilon(x) \\ &= a\left(g_\varepsilon * T\right)\left(\frac{y}{a}\right) - a\left(g_\varepsilon * T\right)(x) + \left(g_\varepsilon * S\right)(x). \end{aligned}$$

Let us estimate the function $y \mapsto \left(g_\varepsilon * T\right)\left(\frac{y}{a}\right)$ on the test function $h \in \mathcal{D}(\mathbb{R})$. By putting $H(z) = h(az)$, we can write:

$$\int \left(g_\varepsilon * T\right)\left(\frac{y}{a}\right) h(y)dy = a\int \left(g_\varepsilon * T\right)(z)\, h(az)dz.$$

Then let us define $\widetilde{T} \in \mathcal{D}'(\mathbb{R})$ by:

$$<\widetilde{T}, h> = \langle aT, [z \mapsto h(az)]\rangle = \langle aT, H\rangle$$

hence:

$$\lim_{\varepsilon\to 0}\int \left(g_\varepsilon * T\right)\left(\frac{y}{a}\right) h(y)dy = \lim_{\varepsilon\to 0} a\int \left(g_\varepsilon * T\right)(z)\, H(z)dz = \langle aT, H\rangle = <\widetilde{T}, h>,$$

then:

$$\lim_{\substack{\varepsilon \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} 0}} \left[y \mapsto \left(g_\varepsilon * T\right)\left(\frac{y}{a}\right)\right] = \widetilde{T}.$$

Then we can write: $[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]$, with :

$$\begin{cases} [w_{\varepsilon,1}] \sim a(1_x \otimes \widetilde{T}_y) \\ [w_{\varepsilon,2}] \sim -a(T_x \otimes 1_y) \\ [w_{\varepsilon,3}] \sim S_x \otimes 1_y \end{cases}$$

and so:

$$u \sim a(1_x \otimes \widetilde{T}_y) - a\left(T_x \otimes 1_y\right) + S_x \otimes 1_y.$$

We can remark that:

$$< \widetilde{\delta}, h >= a\delta \left[z \mapsto h\left(az\right) \right] = ah(0) = a < \delta, h >,$$

so that $\widetilde{\delta} = a\delta$, it follows that, for $T = \delta$, we well rediscover the above result.

$\square$

Chapter 6

A generalized Goursat problem

6.1 Terms of the problem

6.1.1 Problem (P'_G)

We search a solution u to the Goursat problem:

$$\left(P'_G\right)\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u = F(.,.,u)\\[2ex] u_{|(Ox)} = \varphi\\[2ex] u_{|_\gamma} = \psi\end{cases}$$

in an algebra of generalized functions $\mathcal{A}\left(\mathbb{R}^2\right)$ defined in the previous chapter.

We suppose that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under F, that $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}\left(\mathbb{R}^2\right)$ are built on the same ring of generalized constants.

We suppose that the problems:

$$P'_\infty(\varphi_\varepsilon,\psi_\varepsilon)\begin{cases}\dfrac{\partial^2 u_\varepsilon}{\partial x\partial y}(x,y) = F(x,y,u_\varepsilon(x,y))\\[2ex] u_\varepsilon\left(x,0)\right) = \varphi_\varepsilon(x)\\[2ex] u_\varepsilon\left(g(y),y\right) = \psi_\varepsilon(y),\end{cases}$$

have, for every ε, a solution $u_\epsilon \in \mathrm{C}^\infty(\mathbb{R}^2)$.

6.1.2 Giving a meaning to (P'_G)

Giving a meaning to (P'_G) is first giving a meaning to:

$$\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) & (1_G) \\ u_{|(Ox)} = \varphi \in \mathcal{A}(\mathbb{R}) & (2_G) \\ u_{|_\gamma} = \psi \in \mathcal{A}(\mathbb{R}) & (3_G) \end{cases}$$

when $u \in \mathcal{A}\left(\mathbb{R}^2\right)$ and γ is the smooth submanifold of $\mathbb{R}^2$ defined by $x = g(y)$.

Giving a meaning to (1_G), under the hypothesis that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable by F, signifies that for a representative $(u_\varepsilon)_\varepsilon$ of u we must have, for every $(i_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$ and $(j_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$, $\left(\dfrac{\partial^2(u_\varepsilon + i_\varepsilon)}{\partial x \partial y} - F(.,.,u_\varepsilon) + j_\varepsilon\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$.
As: $\left(\dfrac{\partial^2(u_\varepsilon + i_\varepsilon)}{\partial x \partial y} - \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y}\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$ and since: $(F(.,.,u_\varepsilon) + j_\varepsilon - F(.,.,u_\varepsilon))_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$, this comes down to verifying that:

$$\underline{\left(\frac{\partial^2(u_\varepsilon)}{\partial x \partial y} - F(.,.,u_\varepsilon)\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right).}$$

Giving a meaning to (2_G) and (3_G) signifies first defining $u \mid_{(Ox)}$ and $u \mid_\gamma$ and, as γ is a smooth submanifold of $\mathbb{R}^2$ that can be represented by a single map $(\gamma : x = g(y))$, we can identify $\mathcal{A}(\gamma)$ and $\mathcal{A}(\mathbb{R})$ and so, $u \mid_\gamma$ and $u \mid_{(Ox)}$, to the elements of $\mathcal{A}(\mathbb{R})$ some representatives of which are $(y \mapsto u_\varepsilon(g(y), y))_\varepsilon$ and $(x \mapsto u_\varepsilon(x, 0))_\varepsilon$.

So (2_G) is equivalent to:

$(x \mapsto ((u_\varepsilon + i_\varepsilon)(x, 0) - (\varphi_\varepsilon + \alpha_\varepsilon)(x))_\varepsilon \in \mathcal{N}(\mathbb{R})$.

(3_G) is equivalent to:

$(y \mapsto ((u_\varepsilon + i_\varepsilon)(g(y), y) - (\psi_\varepsilon + \beta_\varepsilon)(y)))_\varepsilon \in \mathcal{N}(\mathbb{R})$, for every$(i_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$,$(\alpha_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$,

$(\beta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$, and, considering:

$$(x \mapsto ((u_\varepsilon + i_\varepsilon)(x,0) - (u_\varepsilon(x,0)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(x \mapsto ((\varphi_\varepsilon + \alpha_\varepsilon)(x) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(y \mapsto ((u_\varepsilon + i_\varepsilon)\left(g(y), y\right) - u_\varepsilon\left(g(y), y\right))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(x \mapsto ((\psi_\varepsilon + \beta_\varepsilon)(x) - \psi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(y \mapsto (j_\varepsilon(y) - i_\varepsilon\left(g(y), y\right))_\varepsilon \in \mathcal{N}(\mathbb{R})$$

this comes down to:

$$(x \mapsto (u_\varepsilon(x,0) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

$$(y \mapsto (u_\varepsilon\left(g(y), y\right) - \psi_\varepsilon(y))_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

To sum up, (P'_G) has a meaning if and only if it is represented by a $(u_\varepsilon)_\varepsilon$verifying:

$$\begin{cases} \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y} - F(.,.,u_\varepsilon) \in \mathcal{N}\left(\mathbb{R}^2\right) \\ (x \mapsto (u_\varepsilon(x,0) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}) \\ (y \mapsto (u_\varepsilon\left(g(y), y\right) - \psi_\varepsilon(y))_\varepsilon \in \mathcal{N}(\mathbb{R}). \end{cases}$$

If so, for every ε, the solution u_ε to $P'_\infty\left(\varphi_\varepsilon, \psi_\varepsilon\right)$ is such that $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$ then the relations above are all the more true and $[u_\varepsilon]$ is a solution to (P'_G). $\square$

6.2 Solving the problem

6.2.1 Solving (P'_G)

6.2.1.1. Theorem

Let us suppose that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under F, let us suppose that $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}\left(\mathbb{R}^2\right)$ are built on the same ring $\mathcal{C} = A/I$ of generalized constants. Let us suppose that the data of *problem* (P'_G) *verify the conditions* $\varphi \in \mathcal{A}(\mathbb{R})$, $\psi \in \mathcal{A}(\mathbb{R})$, $g \in \mathrm{C}^\infty(\mathbb{R})$, $\varphi = [\varphi_\varepsilon]$, $\psi = [\psi_\varepsilon]$; $\psi_\varepsilon(0) = \varphi_\varepsilon(g(0))$.

T*hen problem* (P'_G) *has a unique solution* u *in* $\mathcal{A}\left(\mathbb{R}^2\right)$.

Proof.

Let us suppose $g\,(y) \leq x$.

Let $u_\varepsilon = SP'_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ the solution to $P'_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ with the initial conditions $\varphi_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$ and $\psi_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$; that is u_ε verifies the problem:

$$P'_\infty(\varphi_\varepsilon, \psi_\varepsilon)\begin{cases} \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y}(x,y) = F(x,y,u_\varepsilon(x,y)) \\ u_\varepsilon(x,0) = \varphi_\varepsilon(x) \\ u_\varepsilon\,(g(y),y) = \psi_\varepsilon(y). \end{cases}$$

According to the previus result, it is enough to verify that $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$ for $u = [u_\varepsilon]$ to be solution to (P'_G).

Any other solution v to (P'_G) is in the form: $v = [v_\varepsilon]$, where $(v_\varepsilon)_\varepsilon$ verifies:

$$\begin{cases} \dfrac{\partial^2 v_\varepsilon}{\partial x \partial y} - F(.,.,v_\varepsilon) = (i_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right) \\ (x \mapsto (v_\varepsilon(x,0) - \varphi_\varepsilon(x)))_\varepsilon = (\alpha_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}) \\ (y \mapsto (v_\varepsilon\,(g(y),y) - \psi_\varepsilon(y))_\varepsilon = (\beta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}), \end{cases}$$

and so the uniqueness of the solution to (P'_G) will be the consequence of: $(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$

a) Let us show that: $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$.

We will prove that:

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, (P_{K,l}(u_\varepsilon))_\varepsilon \in A_+.$$

Let us proceed by induction showing first that we have:

a1)

$$\forall K \Subset \mathbb{R}^2, (P_{K,0}(u_\varepsilon))_\varepsilon \in A_+$$

with:

$$P_{K,0}(u_\varepsilon) = \sup_K |u_\varepsilon(x)| = \|u_\varepsilon\|_{\infty,K}\,,$$

that is the 0-ordered overestimation is verified.

According to proposition 2.2.1.2., we have: $\forall K \Subset \mathbb{R}^2, \exists K_\lambda \Subset \mathbb{R}^2, K \subset K_\lambda,$

$$\|u_\varepsilon\|_{\infty,K} \leq \|u_\varepsilon\|_{\infty,K_\lambda} \leq \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + \frac{\Phi_{\lambda,\varepsilon}}{m_\lambda} \exp[2\lambda' m_\lambda\,(2\lambda)].$$

Hence: $\left(\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\right)_\varepsilon \in A$ because $[\varphi_\varepsilon]$ and $[\psi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$.

$$m_\lambda = \sup_{(x,y)\in K_\lambda;\ t\in\mathbb{R}} \left|\frac{\partial F}{\partial z}(x,y,t)\right|$$

is a constant which depends entirely on F, K_λ.

$c(K_\lambda) = \dfrac{1}{m_\lambda}\exp[4\lambda' m_\lambda \lambda]$ is a constant which depends entirely on F, g, K_λ.

$\Phi_{\lambda,\varepsilon} = \|F(.,.,0)\|_{\infty,K_\lambda} + m_\lambda \|u_{0,\varepsilon}\|_{\infty,K_\lambda}$ so:

$$\begin{aligned}\frac{\Phi_{\lambda,\varepsilon}}{m_\lambda}\exp[4\lambda' m_\lambda\lambda] &= c(K_\lambda)\Phi_{\lambda,\varepsilon}\\ &= c(K_\lambda)\|F(.,.,0)\|_{\infty,K_\lambda} + \exp[4\lambda' m_\lambda\lambda]\,\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\,.\end{aligned}$$

$c_1(K_\lambda) = c(K_\lambda)\|F(.,.,0)\|_{\infty,K_\lambda}$ is a constant which depends entirely on F, K_λ.

$\exp[4\lambda' m_\lambda\lambda]$ is a constant $c_2(K_\lambda)$ which depends entirely on K_λ, F, g.

Consequently:

$$\|u_\varepsilon\|_{\infty,K} \leq \|u_\varepsilon\|_{\infty,K_\lambda} \leq \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda) + c_2(K_\lambda)\,\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\,,$$

so: $\|u_\varepsilon\|_{\infty,K} \leq \|u_\varepsilon\|_{\infty,K_\lambda} \leq (1 + c_2(K_\lambda))\,\|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda)$.

We have: $\left(\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\right)_\varepsilon \in A$, so $\left((1+c_2(K_\lambda))\,\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\right)_\varepsilon \in A$, (if: $(r_\varepsilon)_\varepsilon \in A$, then: $(cr_\varepsilon)_\varepsilon \in A$) and as $c_1(K_\lambda)$ is a constant ($1 \in A$) we deduce that:

$$((1 + c_2(K_\lambda))\,\|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda))_\varepsilon \in A.$$

A being stabe by overestimation $\left(\|u_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in A$ and so: $\left(\|u_\varepsilon\|_{\infty,K}\right)_\varepsilon \in A$.

a2) Let us show that:

$$\left(P_{K,1}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

We have:

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int_0^y F(x,\eta,u_\varepsilon(x,\eta))d\eta,$$

hence:

$$\begin{aligned}
P_{K,(1,0)}(u_\varepsilon) = \left\|\frac{\partial u_\varepsilon}{\partial x}\right\|_{\infty,K} &= \sup_K \left|\frac{\partial u_\varepsilon}{\partial x}(x,y)\right| \\
&\leq \sup_K \left|\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y)\right| + |y|\left(\sup_{K_\lambda}|F(x,\eta,u_\varepsilon(x,\eta))|\right) \\
&\leq \sup_K \left|\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y)\right| + \lambda\left(\sup_{K_\lambda}|F(x,\eta,u_\varepsilon(x,\eta))|\right).
\end{aligned}$$

As $\mathcal{A}\left(\mathbb{R}^2\right)$ being stable under F there exist C, such that:

$$P_{K_\lambda,(0,0)}(F(.,.,u_\varepsilon)) \leq C P_{K_\lambda,(0,0)}(u_\varepsilon). \qquad (1)$$

We have:

$$\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) = \varphi'_\varepsilon(x),$$

hence: $\left(\left\|\frac{\partial u_{0,\varepsilon}}{\partial x}\right\|_{\infty,K}\right)_\varepsilon \in A$ because $[\varphi_\varepsilon]$ is an element of $\mathcal{A}(\mathbb{R})$.

So:

$$P_{K,(1,0)}(u_\varepsilon) \leq \left\|\frac{\partial u_{0,\varepsilon}}{\partial x}\right\|_{\infty,K} + C\lambda P_{K_\lambda,(0,0)}(u_\varepsilon).$$

A being stable by overestimation: $\left(P_{K,(1,0)}(u_\varepsilon)\right)_\varepsilon \in A$.

We have:

$$\frac{\partial u_\varepsilon}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) + \int_{g(y)}^x F(\xi,y,u_\varepsilon(\xi,y))d\xi - g'(y)\int_0^y F(g(y),\eta,u(g(y),\eta))d\eta,$$

$$\begin{aligned}P_{K,(0,1)}(u_\varepsilon) &= \left\| \frac{\partial u_\varepsilon}{\partial y} \right\|_{\infty,K} = \sup_K \left| \frac{\partial u_\varepsilon}{\partial y}(x,y) \right| \\ &\leq \sup_K \left| \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) \right| + (x - g(y) + |y|\, g'(y)) \left(\sup_{K_\lambda} |F(x,\eta,u_\varepsilon(x,\eta))| \right) \\ &\leq \sup_K \left| \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) \right| + \big(g(\lambda) - g(-\lambda) + \lambda g'(y) \big) \left(\sup_{K_\lambda} |F(x,\eta,u_\varepsilon(x,\eta))| \right).\end{aligned}$$

$\mathcal{A}\left(\mathbb{R}^2\right)$ being stable under F: $\exists C$, $P_{K_\lambda,(0,0)}(F(.,.,u_\varepsilon)) \leq C P_{K_\lambda,(0,0)}(u_\varepsilon)$.

We have:

$$\frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) = \psi'_\varepsilon(y) + g'(y)\varphi'_\varepsilon\left(g\left(y\right)\right),$$

so:

$$\left(\left\| \frac{\partial u_{0,\varepsilon}}{\partial y} \right\|_{\infty,K} \right)_\varepsilon \in A_+,$$

because $[\psi_\varepsilon]$ and $[\varphi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$.

Hence:

$$P_{K,(0,1)}(u_\varepsilon) \leq \left\| \frac{\partial u_{0,\varepsilon}}{\partial y} \right\|_{\infty,K} + C\left(g\left(\lambda\right) - g\left(-\lambda\right) + \lambda g'(y) \right) P_{K_\lambda,(0,0)}(u_\varepsilon)$$

and so, like previously:

$$\left(\left\| \frac{\partial u_\varepsilon}{\partial y} \right\|_{\infty,K} \right)_\varepsilon \in A_+.$$

a3) Induction.

Let us suppose that, for every $l < n$ we have: $\left(P_{K,l}(u_\varepsilon)\right)_\varepsilon \in A_+$ and let us show that involves $\left(P_{K,l+1}(u_\varepsilon)\right)_\varepsilon \in A_+$. In fact we have:

$$P_{K,n+1} = \max\left(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n}\right)$$

with:

$$P_{1,n} = P_{K,(n+1,0)},$$

$$P_{2,n} = P_{K,(0,n+1)},$$

$$P_{3,n} = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)},$$

$$P_{4,n} = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}.$$

a3 1) Let us show first that, for every $n \in \mathbb{N}$,

$$\left(P_{1,n}(u_\varepsilon)\right)_\varepsilon \in A_+,\ \left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

We have:

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}(x,y) = \varphi\text{"}(x) + \int_0^y \frac{\partial}{\partial x} F(x,\eta,u_\varepsilon(x,\eta))d\eta$$

and by successive derivations, for $n \geq 1$:

$$\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n} F(x,\eta,u_\varepsilon(x,\eta))d\eta d\eta,$$

with: $\dfrac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) = \varphi^{(n+1)}(x)$.

As we have taken $K \subset K_\lambda$, we can write:

$$\sup_{(x,y)\in K} \left| \frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) \right| \leq \left\| \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}} \right\|_{\infty,K} + \lambda \left(\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_\varepsilon(x,y)) \right| \right).$$

We have:

$$\left(\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_\varepsilon(x,y)) \right| \right) = P_{K,(n,0)}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)),$$

moreover:

$$\left\| \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}} \right\|_{\infty,K} \in A_+.$$

According to the hypothesis of stability, a simple calculation shows then that, for every $K \Subset \mathbb{R}^2$:

$$\left(P_{K,n}(F(.,.,u_\varepsilon))\right)_\varepsilon \in A_+.$$

Let us show that, for every $n \in \mathbb{N}$, $\left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A_+$.

As we have:

$$\frac{\partial u_\varepsilon}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) + \int_{g(y)}^{x} F(\xi, y, u_\varepsilon(\xi,y))d\xi - g'(y)\int_0^y F(g(y),\eta,u(g(y),\eta))d\eta,$$

and by successive derivations, we deduce that, for $n \geq 1$:

$$\begin{aligned}\frac{\partial^{n+1} u_\varepsilon}{\partial y^{n+1}}(x,y) &= \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial y^{n+1}}(x,y) \\ &\quad - \sum\nolimits_{j=0}^{n-1} C_n^j g^{(n-j)}(y) \frac{\partial^j}{\partial y^j} F(g(y),y,\psi_\varepsilon(y)) - \int_x^{g(y)} \frac{\partial^n}{\partial y^n} F(\xi,y,u_\varepsilon(\xi,y))d\xi \\ &\quad - \sum\nolimits_{j=0}^{n-1} C_n^{j+1} g^{(n-j)}(y) \frac{\partial^j}{\partial y^j} F(g(y),y,\psi_\varepsilon(y)) - g^{(n+1)}(y) \int_0^y F(g(y),\eta,u_\varepsilon(g(y),\eta))d\eta.\end{aligned}$$

As we have taken $K \subset K_\lambda$, we can write:

$$\begin{aligned}&\sup_{(x,y)\in K} \left|\frac{\partial^{n+1} u_\varepsilon}{\partial y^{n+1}}(x,y)\right| \leq \left\|\frac{\partial^{n+1} u_{0,\varepsilon}}{\partial y^{n+1}}\right\|_{\infty,K} + (g(\lambda) - g(\lambda))\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n} F(x,y,u_\varepsilon(x,y))\right|\right) \\ &+ \sup_{y\in[-\lambda,\lambda]} \sum\nolimits_{j=0}^{n-1} C_{n+1}^{j+1}\left|g^{(n-j)}(y)\right| \left|\frac{\partial^j}{\partial y^j}F(g(y),y,\psi_\varepsilon(y))\right| + \lambda g^{(n+1)}(y) \sup_{(x,y)\in K}\left|F(x,y,u_\varepsilon(x,y))\right|.\end{aligned}$$

We have:

$$\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|\right) = P_{K,(0,n)}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)),$$

and, as $\psi_\varepsilon(y) = u_\varepsilon(g\left(y\right),y)$:

$$\begin{aligned}\sup_{y\in[-\lambda,\lambda]}\left|\frac{\partial^j}{\partial y^j}F(g(y),y,\psi_\varepsilon(y))\right| &\leq \left(\sup_{(x,y)\in K}\left|\frac{\partial^i}{\partial y^i}F(x,y,u_\varepsilon(x,y))\right|\right) \\ &\leq P_{K,i}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)).\end{aligned}$$

$$\sup_{(x,y)\in K}\left|F(x,y,u_\varepsilon(x,y))\right| \leq P_{K,1}(F(.,.,u_\varepsilon)).$$

According to the hypothesis of stability, a simple calculation shows then that, for every $K \Subset \mathbb{R}^2$ and every $n \in \mathbb{N}$,

$$\left(P_{K,(0,n+1)}\left(u_\varepsilon\right)\right)_\varepsilon \in A_+.$$

a3 2) For $\alpha + \beta = n$ and $\beta \geq 1$, we have now:

$$\begin{aligned}
P_{K,(\alpha+1,\beta)}(u_\varepsilon) &= \sup_{(x,y)\in K} \left| D^{(\alpha+1,\beta)} u_\varepsilon\left(x,y\right) \right| = \sup_{(x,y)\in K} \left| D^{(\alpha,\beta-1)} D^{(1,1)} u_\varepsilon\left(x,y\right) \right| \\
&= \sup_{(x,y)\in K} \left| D^{(\alpha,\beta-1)} F(x,y,u_\varepsilon\left(x,y\right)) \right| = P_{K,(\alpha,\beta-1)}(F(.,.,u_\varepsilon)) \\
&\leq P_{K,n-1}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)).
\end{aligned}$$

So we finally have:

$$P_{3,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_\varepsilon) \leq P_{K,n}(F(.,.,u_\varepsilon))$$

and the hypothesis of stability then assures that:

$$\left(P_{3,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

In the same way, for $\alpha + \beta = n$ and $\alpha \geq 1$, we have:

$$\begin{aligned}
P_{K,(\alpha,\beta+1)}(u_\varepsilon) &= \sup_{(x,y)\in K} \left| D^{(\alpha,\beta+1)} u_\varepsilon\left(x,y\right) \right| = \sup_{(x,y)\in K} \left| D^{(\alpha-1,\beta)} D^{(1,1)} u_\varepsilon\left(x,y\right) \right| \\
&= \sup_{(x,y)\in K} \left| D^{(\alpha-1,\beta)} F(x,y,u_\varepsilon\left(x,y\right)) \right| = P_{K,(\alpha-1,\beta)}(F(.,.,u_\varepsilon)) \\
&\leq P_{K,n-1}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)).
\end{aligned}$$

So we finally have:

$$P_{4,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\varepsilon) \leq P_{K,n}(F(.,.,u_\varepsilon))$$

and the hypothesis of stability then assures that:

$$\left(P_{4,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

Finally, we clearly have:

$$\left(P_{K,n+1}(u_\varepsilon)\right)_\epsilon \in A_+.$$

So $u = [u_\varepsilon]$ is solution to (P'_G).

b) Let us show that u is the unique solution to (P'_G).

Let $v = [v_\varepsilon]$ an other solution to (P'_G).

There are $(i_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$, $(\alpha_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}\right)$, $(\beta_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}\right)$ such that:

$$\begin{cases} \dfrac{\partial^2 v_\varepsilon}{\partial x \partial y}(x,y) = F(x,y,v_\varepsilon(x,y)) + i_\varepsilon(x,y), \\ v_\varepsilon\left(x,0\right)) = \varphi_\varepsilon(x) + \alpha_\varepsilon(x), \\ \dfrac{\partial v_\varepsilon}{\partial y}\left(g\left(y\right),y\right)) = \psi_\varepsilon(y) + \beta_\varepsilon(x). \end{cases}$$

It is easy to see that:

$$\left(\iint_{D(x,y,g)} i_\varepsilon(\xi,\eta))d\xi d\eta\right)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right).$$

So there is $(j_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^2\right)$ such that:

$$v_\varepsilon(x,y) = v_{0,\varepsilon}(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta + j_\varepsilon(x,y),$$

with: $v_{0,\varepsilon}(x,y) = u_{0,\varepsilon}(x,y) + \theta_\varepsilon(x,y)$, where: $u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(g(y))$ and:

$$\theta_\varepsilon(x,y) = \beta_\varepsilon(y) + \alpha_\varepsilon(x) - \alpha_\varepsilon(g(y))$$

So $(\theta_\varepsilon)_\varepsilon$ belongs to $\mathcal{N}(\mathbb{R}^2)$. So there is $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$

$$v_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \sigma_\varepsilon(x,y) + \iint_{D(x,y,g)} F(\alpha,\beta,v_\varepsilon(\alpha,\beta))d\alpha d\beta.$$

b1) Let us put $w_\varepsilon = v_\varepsilon - u_\varepsilon$ and let us show that: $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$.

We have to prove that:

$$\forall K \Subset \mathbb{R}^2, \forall n \in \mathbb{N}, \left(P_{K,n}(w_\varepsilon)\right)_\varepsilon \in I_A^+.$$

Let us proceed by induction showing first that we have:

$$\left(P_{K,1}\left(w_\varepsilon\right)\right)_\varepsilon \in I_A.$$

We have:

$$w_\varepsilon(x,y) = \iint_{D(x,y,g)} \left(F(\xi,\eta,v_\varepsilon(\xi,\eta)) - F(\xi,\eta,u_\varepsilon(\xi,\eta))\right) d\xi d\eta + \sigma_\varepsilon(x,y),$$

however:

$$F(\xi,\eta,v_\varepsilon(\xi,\eta)) - F(\xi,\eta,u_\varepsilon(\xi,\eta)) = \\ (v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta)) \left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta)))d\theta\right),$$

so:

$$w_\varepsilon(x,y) = -\iint_{D(x,y,g)} w_\varepsilon(\xi,\eta) \left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(w_\varepsilon(\xi,\eta)))d\theta\right) d\xi d\eta + \sigma_\varepsilon(x,y). \quad (3)$$

Let $(x,y) \in K_\lambda$, since $D(x,y,g) \subset K_\lambda$, if $g\,(y) \leq x$, we have:

$$\begin{aligned} |w_\varepsilon(x,y)| &\leq m_\lambda \int_{g(y)}^{x} \int_0^y |w_\varepsilon(\xi,\eta)|\, d\xi d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda} \\ &\leq m_\lambda \int_{-g(\lambda)}^{+g(\lambda)} \int_0^y |w_\varepsilon(\xi,\eta)|\, d\xi d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda}. \end{aligned}$$

Let us put: $e_\varepsilon(y) = \sup\limits_{\xi \in [g(-\lambda);g(\lambda)]} |w_\varepsilon(\xi,y)|$, then:

$$|w_\varepsilon(x,y)| \leq m_\lambda 2\lambda' \int_0^y e_\varepsilon(\eta)d\eta + \|\sigma_\varepsilon\|_{\infty,k_\lambda},$$

we deduce that, for every $y \in [0;\lambda]$, if $g\,(y) \leq x$,

$$e_\varepsilon(y) \leq m_\lambda 2\lambda' \int_0^y e_\varepsilon(\eta)d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

Thus according to the Gronwall lemma, for every $y \in [0;\lambda]$, if $g(y) \leq x$,

$$e_\varepsilon(y) \leq \left(\exp(\int_0^y m_\lambda 2\lambda d\eta)\right) \|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

For every $y \in [0;\lambda]$, if $g(y) \leq x$,

$$e_\varepsilon(y) \leq \left(\exp(m_\lambda 2\lambda' y)\right) \|\sigma_\varepsilon\|_{\infty,K_\lambda} \leq \left(\exp(m_\lambda 2\lambda' \lambda)\right) \|\sigma_\varepsilon\|_{\infty,K_\lambda} \leq \left(\exp(m_\lambda 2\lambda' \lambda)\right) \|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

We obtain the same result in the other cases, hence:

$$\forall y \in [-\lambda;\lambda], \, e_\varepsilon(y) \leq \|\sigma_\varepsilon\|_{\infty,K_\lambda} \left(\exp(m_\lambda 2\lambda' \lambda)\right),$$

consequently:

$$\|w_\epsilon\|_{\infty,K_\lambda} \leq \|\sigma_\varepsilon\|_{\infty,K_\lambda} \left(\exp(m_\lambda 2\lambda' \lambda)\right.,$$

$(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ so $\left(\|\sigma_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in I_A$.

$(\exp(m_\lambda 2\lambda' \lambda)) \|\sigma_\varepsilon\|_{\infty,K_\lambda}$ is a constant, consequently $\left(\|w_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in I_A$.

Which involves the 0-ordered estimation.

b2) Induction.

Let us suppose that, for every $l \leq n$, we have: $(P_{K,l}(w_\varepsilon))_\varepsilon \in I_A^+$ and let us show that involves $(P_{K,n+1}(w_\varepsilon))_\varepsilon \in I_A^+$.

b21) First, let us show that, for every $n \in \mathbb{N}$:

$$(P_{1,n}(w_\varepsilon))_\varepsilon \in I_A^+.$$

We have:

$$\frac{\partial w_\varepsilon}{\partial x}(x,y) = \frac{\partial \sigma_\varepsilon}{\partial x}(x,y) + \int_0^y \left(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta))\right) d\eta$$

and by successive derivations, for $n \geq 1$:

$$\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n} F(x,\eta,u_\varepsilon(x,\eta)) d\eta,$$

so:

$$\frac{\partial^{n+1} w_\varepsilon}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} \sigma_\varepsilon}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n}\left(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta)\right) d\eta.$$

Hence:

$$\begin{aligned} P_{K,(n+1,0)}(w_\varepsilon) \leq P_{K,(n+1,0)}(\sigma_\varepsilon) + \\ + (\lambda)\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}\left(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta))\right|\right).\end{aligned}$$

We have:

$$\begin{aligned}\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}\left(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta))\right|\right) &= P_{K,(n,0)}\left(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)\right) \\ &\leq P_{K,n}\left(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)\right).\end{aligned}$$

According to the hypothesis of stability, for every $K \Subset \mathbb{R}^2$:

$$\left(P_{K,(n+1,0)}(w_\varepsilon)\right)_\epsilon \in I_A^+.$$

Let us show that, for every $n \in \mathbb{N}$, $\left(P_{2,n}(w_\varepsilon)\right)_\varepsilon \in I_A^+$.

We have:

$$\begin{aligned}\frac{\partial^{n+1} u_\varepsilon}{\partial y^{n+1}}(x,y) &= \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial y^{n+1}}(x,y) - \int_x^{g(y)} \frac{\partial^n}{\partial y^n} F(\xi,y,u_\varepsilon(\xi,y))d\xi \\ &- \sum\nolimits_{j=0}^{n-1} C_{n+1}^{j+1} g^{(n-j)}(y)\frac{\partial^j}{\partial y^j}F(g(y),y,\psi_\varepsilon(y)) - g^{(n+1)}(y)\int_0^y F(g(y),\eta,u_\varepsilon(g(y),\eta))d\eta,\end{aligned}$$

so:

$$\begin{aligned}\frac{\partial^{n+1} w_\varepsilon}{\partial y^{n+1}}(x,y) &= \frac{\partial^{n+1} \sigma_\varepsilon}{\partial y^{n+1}}(x,y) + \mu_\varepsilon(y) - \int_x^{g(y)}\left(\frac{\partial^n}{\partial y^n}F(x,y,v_\varepsilon(x,y)) - \frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right)d\xi \\ &- g^{(n+1)}(y)\int_0^y \left(F(g(y),\eta,v_\varepsilon(g(y),\eta)) - F(g(y),\eta,u_\varepsilon(g(y),\eta))\right)d\eta,\end{aligned}$$

with:

$$\mu_\varepsilon(y) = \sum\nolimits_{j=0}^{n-1} C_{n+1}^{j+1} g^{(n-j)}(y)\left(\frac{\partial^j}{\partial y^j}F(g(y),y,\psi_\varepsilon(y)) - \frac{\partial^j}{\partial y^j}F(g(y),y,\psi_\varepsilon(y)+\beta_\varepsilon(y))\right).$$

$\left(\mu_\varepsilon\right)_\varepsilon \in \mathcal{N}(\mathbb{R})$. Hence:

$$\begin{aligned}
P_{K,(0,n+1)}(w_\varepsilon) \leq P_{K,(0,n+1)}(\sigma_\varepsilon) &+ \sup_{y\in[-\lambda,\lambda]} |\mu_\varepsilon(y)| \\
&+ (g(\lambda)-g(-\lambda))\left(\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,v_\varepsilon(x,y)) - \frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|\right) \\
&+ \lambda g^{(n+1)}(y)\left(\sup_{(x,y)\in K}|F(x,y,v_\varepsilon(x,y)) - F(x,y,u_\varepsilon(x,y))|\right).
\end{aligned}$$

We have:

$$\begin{aligned}
\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,v_\varepsilon(x,y)) - \frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right| &= P_{K,(0,n)}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \\
&\leq P_{K,(0,n)}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)).
\end{aligned}$$

According to the hypothesis of stability, for every $K \Subset \mathbb{R}^2$: $\left(P_{K,(0,n+1)}(w_\varepsilon)\right)_\epsilon \in I_A$.

b2 2) For $\alpha+\beta = n$ and $\beta \geq 1$, we have:

$$P_{K,(\alpha+1,\beta)}(w_\varepsilon) = P_{K,(\alpha,\beta-1)}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)).$$

So we finally have:

$$P_{3,n}(w_\varepsilon) = \sup_{\alpha+\beta=n,\beta\geq 1} P_{K,(\alpha+1,\beta)}(w_\varepsilon) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon))$$

and the hypothesis of stability then assures that:

$$\left(P_{3,n}(w_\varepsilon)\right)_\varepsilon \in I_A^+.$$

For $\alpha+\beta = n$ and $\alpha \geq 1$, we have now:

$$P_{K,(\alpha,\beta+1)}(w_\varepsilon) = P_{K,(\alpha-1,\beta)}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon)).$$

So we finally have:

$$P_{4,n}(w_\varepsilon) = \sup_{\alpha+\beta=n,\alpha\geq 1} P_{K,(\alpha,\beta+1)}(w_\varepsilon) \leq P_{K,n-1}(F(.,.,v_\varepsilon) - F(.,.,u_\varepsilon))$$

and the hypothesis of stability then assures that:

$$\left(P_{4,n}(w_\varepsilon)\right)_\varepsilon \in I_A^+.$$

So for every $l \leq n+1$, we have:

$$\left(P_{K,l}(w_\varepsilon)\right)_\epsilon \in I_A^+.$$

Proceeding by induction we obtain, for every $n \in \mathbb{N}$:

$$\left(P_{K,n}(w_\varepsilon)\right)_\epsilon \in I_A^+.$$

So $(w_\epsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$; consequently u is the unique solution to (P'_G). $\square$

6.3 A (degenerate) Goursat problem in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

6.3.1 Terms of the problem

We search a generalized solution u to the following Goursat problem with irregular data:

$$(P'_G)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|(Ox)} = \varphi_\varepsilon \\ u_{|(Oy)} = \psi_\varepsilon \end{cases}$$

where φ and ψ are one-variable generalized functions. The notation $F(.,.,u)$ extend, with a meaning above defined, the expression $(x,y) \mapsto F(x,y,u(x,y))$ to the case where u is a generalized function of two variables x and y.

In all cases the following hypothesis will be satisfied:

$$(H): \begin{cases} F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R}) \\ \forall K \Subset \mathbb{R}^2, \ \sup\limits_{\substack{(x,y)\in K \\ z \in \mathbb{R}}} |\partial_z F(x,y,z)| < +\infty \end{cases}$$

where the notation $K \Subset \mathbb{R}^2$ means that K is a compact of $\mathbb{R}^2$.

Hypothesis (H) being satisfied, $\mathcal{A}(\mathbb{R}^2)$ being stable under F. If the data of problem (P_G') verify the conditions:

$$\varphi \in \mathcal{A}\left(\mathbb{R}\right), \, \psi \in \mathcal{A}\left(\mathbb{R}\right), \, g(y) = 0,$$

the problem has a unique solution $[u_\varepsilon] \in \mathcal{A}(\mathbb{R}^2)$.

$$u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \iint_{D(x,y,0)} F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta;$$

$$u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon\left(0\right).$$

$$u_{\varepsilon,n}(x,y) = u_{0,\varepsilon}(x,y) + \iint_{D(x,y,0)} F(\xi,\eta,u_{\varepsilon,n-1}(\xi,\eta))d\xi d\eta, \, n \geq 1.$$

6.3.2 Solving the problem

6.3.2. Theorem

The generalized solution u to the following Goursat problem:

$$\left(P_G'\right) \begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|(Ox)} = \varphi \\ u_{|(Oy)} = \psi \end{cases}$$

where φ *and* ψ *are one-variable generalized functions, is* $u = [u_\varepsilon]$ with:

$$u_\varepsilon = \lim_{n \to +\infty} u_{\varepsilon,n} \text{ and } u_{\varepsilon,n}\left(x,y\right) = u_{0,\varepsilon}(x,y) + \int_0^x \left(\int_0^y F(\xi,\eta,u_{\varepsilon,n-1}(\xi,\eta))d\eta \right) d\xi;$$

$$u_{0,\varepsilon}(x,y) = \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \varphi_\varepsilon(0).$$

(We take $g = 0$). $\square$

6.3.3. Corollary

Then we have:

$$u_{\varepsilon}\left(x,y\right)=u_{0,\varepsilon}(x,y)+\int_{0}^{x}\left(\int_{0}^{y}F(\xi,\eta,u_{\varepsilon}(\xi,\eta))d\eta\right)d\xi.$$

□

Chapter 7

Qualitative study of the solution

7.1 Parametric singular spectrum of the solution to the Goursat problem

7.1.1 Relation between the $\mathcal{D}'$-parametric singular spectrum of solution u and the $\mathcal{D}'$-parametric singular spectrum of u_0

7.1.1.1. Theorem

We put $u_0 = [u_{0,\varepsilon}]$ *with*

$$u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(g(y)),$$

and we suppose that:

$$(H_2) \quad \forall K \Subset \mathbb{R}^2, \mathcal{M}_F(K) = \sup_{(x,y)\in K, z\in\mathbb{R}} |F(x,y,z)| < +\infty.$$

Then the restriction to the parametric singular support of u_0 *of the* $\mathcal{D}'$*-parametric singular spectrum of the solution* u *to the Goursat problem* (P'_G) *is inclued in the restriction to the parametric singular support of* u_0 *of the* $\mathcal{D}'$*-parametric singular spectrum of* u_0*. In other words, over the singular support of* u_0*, there is no increase in the distributionnal* *singularities of* u *in comparison with those of* u_0.

Proof.

Let $(x_0, y_0) = X \in S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0$ and $r \in N_{\mathcal{D}',X}(u_0)$. It results from the definitions that we have: $\Sigma_{\mathcal{D}',X}(u_0) \neq \emptyset$, and so that $N_{\mathcal{D}',X}(u_0) \subset]0, +\infty[$ which involves that $r > 0$.

Let us show that we then have: $r \in N_{\mathcal{D}',X}(u)$.

From the of $N_{\mathcal{D}',X}(u_0)$, there exists a neighbourhood V_X of X such that:

$$\lim_{\varepsilon \to 0} (\varepsilon^r u_{\varepsilon|_{V_X}}) \in \mathcal{D}'(V_X).$$

Let $f \in \mathcal{D}(V_X)$. So, there exists some distribution $T \in \mathcal{D}'(V_X)$ such that:

$$\lim_{\varepsilon \to 0} \iint_{V_X} \varepsilon^r u_{0,\varepsilon}(x,y) f(x,y) dxdy = T(f).$$

Let us show that:

$$\iint_{V_X} \varepsilon^r \left[u_\varepsilon(x,y) - u_{0,\varepsilon}(x,y) \right] f(x,y) dxdy$$

has 0 for limit when ε tends to 0.

Supposing moreover that $g(y) \leq x$.

As $u_\varepsilon(x,y) - u_{0,\varepsilon}(x,y) = -\iint_{D(x,y,g)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\xi d\eta$

and that (with the above notations):

$$\begin{aligned}
&\left| \iint_{V_X} \left[\iint_{D(x,y,g)} F(\xi, \eta, u_\varepsilon(\xi, \eta)) d\xi d\eta \right] f(x,y) dxdy \right| \\
&\qquad \leq \mathcal{M}_F(suppf) \left| \iint_{suppf} \left[\iint_{D(x,y,f)} d\xi d\eta \right] f(x,y) dxdy \right| \\
&\qquad \leq \mathcal{M}_F(suppf) \left| \iint_{suppf} (A(x,y))\, f(x,y) dxdy \right| \\
&\qquad \leq \mathcal{M}_F(suppf) \left| \iint_{suppf} (2\lambda\, |y|)\, f(x,y) dxdy \right| \\
&\qquad \leq 2\lambda \mathcal{M}_F(suppf) \iint_{suppf} |y|\, |f(x,y)|\, dxdy < +\infty,
\end{aligned}$$

putting always $2\lambda' = g(\lambda) - g(-\lambda)$, then we have:

$$\begin{aligned}\limsup_{\varepsilon\to 0}\left|\iint_{V_X}\varepsilon^r\left[u_\varepsilon(x,y)-u_{0,\varepsilon}(x,y)\right]f(x,y)dxdy\right| \\ \leq \limsup_{\varepsilon\to 0}\varepsilon^r\left|\iint_{V_X}\left[\iint_{D(x,y,g)}F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta\right]f(x,y)dxdy\right| \\ \leq \limsup_{\varepsilon\to 0}\varepsilon^r\left[2\lambda'\left(\mathcal{M}_F(suppf)\right)\iint_{suppf}|y|\left|f(x,y)\right|dxdy\right]=0,\end{aligned}$$

because $r \neq 0$. And so: $\lim\limits_{\varepsilon\to 0}\iint_{V_X}\varepsilon^r u_\varepsilon(x,y)f(x,y)dxdy = \lim\limits_{\varepsilon\to 0}\iint_{V_X}\varepsilon^r u_{0,\varepsilon}(x,y)f(x,y)dxdy = T(f)$.

It follows that:

$$\lim_{\varepsilon\to 0}(\varepsilon^r u_{\varepsilon|_{V_X}}) = \lim_{\varepsilon\to 0}(\varepsilon^r u_{0,\varepsilon|_{V_X}}) \in \mathcal{D}'(V_X).$$

So we have: $r \in N_{\mathcal{D}',X}(u)$, which proves the inclusion: $N_{\mathcal{D}',X}(u_0) \subset N_{\mathcal{D}',X}(u)$, and consequently, $\Sigma_{\mathcal{D}',X}(u) \subset \Sigma_{\mathcal{D}',X}(u_0)$, then we have:

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u_{S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u_0} \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u_{0\,S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}u_0}.$$

□

7.1.2 Examples

Let us take $f \in \mathcal{D}(\mathbb{R})$, $f \geq 0$, $\int_{\mathbb{R}} f(x)dx = 1$.

With the above notations we have:

$\varphi = [\varphi_\varepsilon]_\varepsilon$ and $\psi = [\psi_\varepsilon]_\varepsilon$; $u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(g(y))$.

$g(y) = \frac{y}{a}$, $a > 0$.

Let us consider the following cases:

1) $\psi_\varepsilon(y) = \varepsilon^{-1}f(y\varepsilon^{-1})$, $\varphi_\varepsilon(x) = \varepsilon^{-1}f(x\varepsilon^{-1})$ so:

$$\varphi_\varepsilon(g(y)) = \varepsilon^{-1}f((g(y)\varepsilon^{-1}) = \varepsilon^{-1}f(y(a\varepsilon)^{-1}) = a(\frac{1}{a\varepsilon})f(\frac{y}{a\varepsilon}),$$

$$u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(g(y)) = \varepsilon^{-1}f(y\varepsilon^{-1}) + \varepsilon^{-1}f(x\varepsilon^{-1}) - a(\frac{1}{a\varepsilon})f(\frac{y}{a\varepsilon}).$$

$N_{\mathcal{D}',X}(u_0) = [1, +\infty[$, then we have: $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0, 1[$.

2) $\psi_\varepsilon(y) = \varepsilon^{-1} f(y\varepsilon^{-1})$ and:

$$\varphi_\varepsilon(x) = \varepsilon^{-2} f(x\varepsilon^{-1}) = \varepsilon^{-1} \left[\varepsilon^{-1} f(x\varepsilon^{-1})\right],$$

$$\varphi_\varepsilon(g(y)) = \varepsilon^{-2} f((g(y)\varepsilon^{-1}) = \varepsilon^{-2} f(y(a\varepsilon)^{-1}) = a(\frac{1}{(a\varepsilon)^2}) f(\frac{y}{a\varepsilon}),$$

so:

$N_{\mathcal{D}',X}(u_0) = [2, +\infty[$, then we have: $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0, 2[$.

3) $\psi_\varepsilon(y) = f(y\varepsilon^{-1})$ and:

$$\varphi_\varepsilon(x) = f(x\varepsilon^{-1}) = \varepsilon[\varepsilon^{-1} f(x\varepsilon^{-1})],$$

$$\varphi_\varepsilon(g(y)) = \varepsilon[\varepsilon^{-1} f((g(y)\varepsilon^{-1})] = \varepsilon[\varepsilon^{-1} f(y(a\varepsilon)^{-1})] = a\varepsilon[(\frac{1}{(a\varepsilon)}) f(\frac{y}{a\varepsilon})].$$

$N_{\mathcal{D}',X}(u_0) = [0, +\infty[$, then we have: $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times \varnothing$.

$\square$

7.2 Qualitative study of the solution. Case: $F = 0$

7.2.1 Terms of the problem

We search for a generalized solution u to the following problem:

$$(P'_G) \begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|(Ox)} = \varphi \\ u_{|_\gamma} = \psi \end{cases}$$

considering as data the curve γ of equation $x = g(y)$.

Let:

$$P'_\infty(\varphi_\varepsilon,\psi_\varepsilon)\begin{cases}\dfrac{\partial^2 u_\varepsilon}{\partial x\partial y}(x,y)=0\\[2ex] u_\varepsilon(x,0)=\varphi_\varepsilon(x)\\[2ex] u_\varepsilon\left(g(y),y\right)=\psi_\varepsilon(y)\end{cases}$$

We the above notations, we have:

$$\varphi=[\varphi_\varepsilon]\,,\ \psi=[\psi_\varepsilon]\ \text{ and } u_\varepsilon(x,y)=\psi_\varepsilon(y)+\varphi_\varepsilon(x)-\varphi_\varepsilon(g(y)).$$

□

7.2.2 Qualitative study of the solution. Case: $F=0$, $g(y)=\frac{y}{a}$, $(a>0)$

Remark

Then we have: $\psi(0)=\varphi(g(0))$, however: $g(0)=0$, consequently: $\psi(0)=\varphi(0)$.

Case: $g(y)=\frac{y}{a}$, $(a>0)$**,** $\varphi\sim\delta$**,** $\psi\sim\delta$**,**

the association is defined in the following way:

let us consider $f\in\mathcal{D}(\mathbb{R})$, verifying $\int_{\mathbb{R}}f(\xi)d\xi=1$. Let us put $\varphi_\varepsilon(x)=\frac{1}{\varepsilon}f(\frac{x}{\varepsilon}),\psi_\varepsilon(y)=\frac{1}{\varepsilon}f(\frac{y}{\varepsilon})$. Then $(\varphi_\varepsilon)_\varepsilon$ have, in a distributional sense, δ_x as limit and $(\psi_\varepsilon)_\varepsilon$ have δ_y as limit. So $\varphi=[\varphi_\varepsilon]$ is associated to δ_x and $\psi=[\psi_\varepsilon]$ is associated to δ_y.

The solution w to $P'_\infty(\varphi_\varepsilon,\psi_\varepsilon)$ is defined by:

$$w_\varepsilon(x,y)=\varphi_\varepsilon(x)+\psi_\varepsilon(y)-\varphi_\varepsilon(g(y));$$

$$w_\varepsilon=w_{\varepsilon,1}+w_{\varepsilon,2}+w_{\varepsilon,3}.$$

If φ_ε and ψ_ε are mollifiers, we have:

$$\lim_{\substack{\mathcal{D}'(\mathbb{R})\\ \varepsilon\longrightarrow 0}}\varphi_\varepsilon=\delta_x \text{ and } \lim_{\substack{\mathcal{D}'(\mathbb{R})\\ \varepsilon\longrightarrow 0}}\psi_\varepsilon=\delta_y,$$

$$\varphi_\varepsilon(g(y)) = \varepsilon^{-1}f((g(y)\varepsilon^{-1}) = \varepsilon^{-1}f(y(a\varepsilon)^{-1}) = a(\frac{1}{a\varepsilon})f(\frac{y}{a\varepsilon}),$$

$$\psi_\varepsilon(y) = \varepsilon^{-1}f(y\varepsilon^{-1}) = (\frac{1}{\varepsilon})f(\frac{y}{\varepsilon}).$$

Then we have:

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]\,,$$

with:

$$\begin{cases} [w_{\varepsilon,1}] \sim \delta_x \otimes 1_y \\ [w_{\varepsilon,2}] \sim (1_x \otimes \delta_y) \\ [w_{\varepsilon,3}] \sim -a(1_x \otimes \delta_y). \end{cases}$$

□

Case: $g(y) = \frac{y}{a}$**,** $(a > 0)$**,** $\varphi \sim S$**,** $\psi \sim T$**;** $S \in \mathcal{D}'(\mathbb{R})$**,** $T \in \mathcal{D}'(\mathbb{R})$,

the association being achieved choosing

$$\varphi = [f_\varepsilon * S] \text{ and } \Psi = [f_\varepsilon * T]\,,$$

since

$$\lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} (f_\varepsilon * S)_\varepsilon = S \text{ and } \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} (f_\varepsilon * T)_\varepsilon = T.$$

Then we have here:

$$\begin{aligned} u_\varepsilon(x,y) &= \varphi_\varepsilon(x) + \psi_\varepsilon\,(y) - \varphi_\varepsilon(g\,(y)) \\ &= (f_\varepsilon * S)\,(x) + (f_\varepsilon * T)\,(y) - (f_\varepsilon * S)\left(\frac{y}{a}\right). \end{aligned}$$

Let us define $\widetilde{S} \in \mathcal{D}'(\mathbb{R})$ by:

$$< \widetilde{S}, h >= \langle aS, [z \mapsto h\,(az)] \rangle = \langle aS, H \rangle\,.$$

Hence:

$$\lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} \left[y \mapsto (f_\varepsilon * S)\left(\frac{y}{a}\right) \right] = \widetilde{S}.$$

Then we can write: $[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]$, with :

$$\begin{cases} [w_{\varepsilon,1}] \sim S_x \otimes 1_y \\ [w_{\varepsilon,2}] \sim 1_x \otimes T_y \\ [w_{\varepsilon,3}] \sim -\left(1_x \otimes \widetilde{S}_y\right) \end{cases}$$

and so:

$$u \sim S_x \otimes 1_y + 1_x \otimes T_y - \left(1_x \otimes \widetilde{S}_y\right).$$

As $\widetilde{\delta} = a\delta$, it follows that, for $S = \delta$, we clearly rediscover the above result.

□

Part IV

Characteristic problems

Chapter 8

A characteristic Cauchy problem in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

8.1 Problem (P_c)

8.1.1 Terms of the problem

The characteristic Cauchy problem:

$$(P_C)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|(Ox)} = \varphi \\ \dfrac{\partial u}{\partial y} \mid_{(Ox)} = \psi, \end{cases}$$

has no smooth solution (not even C^2) even if the data φ and ψ are smooth too.

Then we can approach it by a family of non-characteristic problems $(P_\varepsilon)_\varepsilon$

$$P_\varepsilon\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = F(.,.,u) \\ u_{|\gamma_\varepsilon} = \varphi \\ \dfrac{\partial u}{\partial y} \mid_{\gamma_\varepsilon} = \psi, \end{cases}$$

by considering the straight line γ_ε of equation $y=\varepsilon x$ as data. We try to give a meaning to the family of solutions by putting it in terms of generalized function which belongs to an algebra appropriately defined.

8.2 Case of regular data

8.2.1.1. Notations, reminders and hypothesis

Rewriting the solution to P_ε, we replace $f\left(x\right)$ by εx and K_λ by $\left[-\frac{a}{\varepsilon};+\frac{a}{\varepsilon}\right]\times[-a;+a]$.

Here we have:

$$u_\varepsilon(x,y)=u_{0,\varepsilon}(x,y)-\iint_{D_\varepsilon(x,y)}F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta,$$

where: $u_{0,\varepsilon}(x,y)=\varphi(x)-\varepsilon\Psi\left(x\right)+\varepsilon\Psi\left(\frac{y}{\varepsilon}\right)$

and:

1) Ψ is a primitive of ψ,

2)

$$D_\varepsilon(x,y)=\begin{cases}\left\{(\xi,\eta)/x\leq\xi\leq\frac{y}{\varepsilon},\varepsilon\xi\leq\eta\leq y\right\}, \text{ if } y\geq\varepsilon x,\\ \left\{(\xi,\eta)/\frac{y}{\varepsilon}\leq\xi\leq x,y\leq\eta\leq\varepsilon\xi\right\}, \text{ if } y\leq\varepsilon x.\end{cases}$$

We put:

$$\begin{aligned}K_\varepsilon&=\left[-\frac{a}{\varepsilon},\frac{a}{\varepsilon}\right]\times[-a,a],\\ m_\varepsilon&=\sup_{(\xi,\eta)\in K_\varepsilon;\ t\in\mathbb{R}}\left|\frac{\partial F}{\partial z}(\xi,\eta,t)\right|,\\ \Phi_\varepsilon&=\sup_{K_\varepsilon}\left|F(x,y,0)\right|+m_\varepsilon\left\|u_{0,\varepsilon}\right\|_{\infty,K_\varepsilon}.\end{aligned}$$

We make the following hypothesis:

$$(H_1)\left\{\begin{array}{l}\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \exists m(K,l), \max\limits_{\alpha\in\mathbb{N}^3,|\alpha|\leq l}\left(\sup\limits_{(x,y)\in K;\ z\in\mathbb{R}}|D^\alpha F(x,y,z)|\right) \leq m(K,l) \\ \qquad\qquad \exists \left(M_\varepsilon\right)_\varepsilon \in \mathbb{R}_*^{]0,1]}, \exists C(l) \in \mathbb{R}_+^*, m(K_\varepsilon,l) \leq C(l)M_\varepsilon\end{array}\right.$$

$$(H2)\left\{\begin{array}{l}\exists \left(r_\varepsilon\right)_\varepsilon \in \mathbb{R}_*^{]0,1]} \text{ such that } \forall K_2 \Subset \mathbb{R}, \forall \alpha_2 \in \mathbb{N}, \exists D_2 \in \mathbb{R}_+^*, \exists p \in \mathbb{N}, \\ \qquad \max\left[\sup\limits_{K_2}\left|D^{\alpha_2}\varphi(\tfrac{y}{\varepsilon})\right|, \sup\limits_{K_2}\left|D^{\alpha_2}\Psi(\tfrac{y}{\varepsilon})\right|\right] \leq \frac{D_2}{(r_\varepsilon)^p}\end{array}\right.$$

$$(H_3)\left\{\begin{array}{l}\mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{]0,1]}\text{:} \\ \qquad\qquad (\varepsilon)_\varepsilon\,; (r_\varepsilon)_\varepsilon\,; \left(e^{\frac{m_\varepsilon}{\varepsilon}}\right)_\varepsilon ; (M_\varepsilon)_\varepsilon\,.\end{array}\right.$$

$$(H_4)\left\{\begin{array}{l}\quad \mathcal{A}\left(\mathbb{R}^2\right) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2) \text{ is built on } \mathcal{C} \text{ with} \\ \qquad (\mathcal{E},\mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}^2), (P_{K,l})_{K\Subset\mathbb{R}^2, l\in\mathbb{N}}\right) \\ \text{and } \mathcal{A}\left(\mathbb{R}^2\right) \text{ is stable under } F \text{ relatively to } \mathcal{C}.\end{array}\right.$$

8.2.1.2. Theorem

With the notations and the hypothesis of the above paragraph 8.2.1.1., if u_ε *is the solution to the problem* (P_ε)*, the family* $(u_\varepsilon)_\varepsilon$ *is the representative of a generalized function which belongs to algebra* $\mathcal{A}(\mathbb{R}^2)$.

Proof.

We have:

$$u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) - \iint_{D_\varepsilon(x,y)} F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta = u_{0,\varepsilon}(x,y) - u_{1,\varepsilon}(x,y),$$

where:

$$u_{0,\varepsilon}(x,y) = \varphi(x) - \varepsilon\Psi\,(x) + \varepsilon\Psi\left(\frac{y}{\varepsilon}\right),$$

Ψ being a primitive of ψ, and:

$$u_{1,\varepsilon}(x,y) = \iint_{D_\varepsilon(x,y)} F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta.$$

a) For $K = K_1 \times K_2 = [-a;a] \times [-a;a]$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, there exist $C_1 > 0$ and $C_2 > 0$ such that:

$$\sup_{K_1} |D^{\alpha_1}\varphi(x)| \leq C_1 (K_1, \alpha_1);$$

$$\varepsilon \sup_{K_1} |D^{\alpha_1}\Psi(x)| \leq \varepsilon C_2 (K_1, \alpha 1).$$

$G(y) = \Psi \circ f_\varepsilon^{-1}(y) = \Psi\left(\frac{y}{\varepsilon}\right)$, we can write:

$$\varepsilon \sup_{K_2} |D^{\alpha_2} G(y)| \leq \frac{D_2}{\varepsilon^{\alpha_2 - 1} (r_\varepsilon)^{p(\alpha_2, K_2)}},$$

so $(P_{K,\alpha}(u_{0,\varepsilon}))_\varepsilon \in A_+$.

b) We have to show that: $(P_{K,\alpha}(u_\varepsilon))_\varepsilon \in A_+$.

Now we have:

$$u_{1,\varepsilon}(x,y) = \iint_{D_\varepsilon(x,y)} F(\xi, \eta, u_\varepsilon(\xi,\eta)) d\xi d\eta.$$

According to the above results:

$$\sup_K \left| \iint_{D_\varepsilon(x,y)} F(\xi, \eta, u_\varepsilon(\xi,\eta)) d\xi d\eta \right| \leq \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda m_\lambda (f(\lambda) - f(-\lambda))],$$

with: $f(x) = \varepsilon x$; $\lambda = \dfrac{a}{\varepsilon}$; $m_\lambda = m_\varepsilon$; so $(f(\lambda) - f(-\lambda)) = 2a$ and

$$2\lambda m_\lambda (f(\lambda) - f(-\lambda)) = 2\frac{a}{\varepsilon} 2a m_\varepsilon = 4\frac{a^2}{\varepsilon} m_\varepsilon,$$

hence:

$$\sup_{K\varepsilon} |u_{1,\varepsilon}(x,y)| \leq \frac{\Phi_\varepsilon}{m_\varepsilon} e^{\frac{4a^2}{\varepsilon} m_\varepsilon},$$

with:

$$\Phi_\varepsilon = \sup_{K\varepsilon} |F(x,y,0)| + m_\varepsilon \|u_{0,\varepsilon}\|_{\infty, K_\varepsilon} \leq C(0) M_\varepsilon + m_\varepsilon \left(\frac{3D_2}{(r_\varepsilon)^{p1}} \right),$$

where: $p1 = p([-a,a], 0)$.

So $\left(P_{K,0}\left(u_{1,\varepsilon}\right)\right)_{\varepsilon} \in A_{+}$, hence: $\left(P_{K,0}\left(u_{\varepsilon}\right)\right)_{\varepsilon} \in A_{+}$.

Moreover:

$$\frac{\partial u_{\varepsilon}}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int_{f(x)}^{y} F(x,\eta,u_{\varepsilon}(x,\eta))d\eta.$$

We have:

$$\frac{\partial u_{1,\varepsilon}}{\partial x} = \int_{f(x)}^{y} F(x,\eta,u_{\varepsilon}(x,\eta))d\eta,$$

so, according to hypothesis (H_1):

$$\sup_{K_{\varepsilon}}\left(\int_{f(x)}^{y}\left|F(x,\eta,u_{\varepsilon}(x,\eta))\right|d\eta\right) \leq 2a\left(m(K_{\varepsilon},0)\right),$$

so: $\left(P_{K,(1,0)}(u_{1,\varepsilon})\right)_{\varepsilon} \in A_{+}$, hence $\left(P_{K,(1,0)}(u_{\varepsilon})\right)_{\varepsilon} \in A_{+}$.

We have:

$$\frac{\partial u_{\varepsilon}}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) - \int_{x}^{f^{-1}(y)} F(\xi,y,u_{\varepsilon}(\xi,y))d\xi.$$

In the same way , we get:

$$\begin{aligned}\sup_{K_{\varepsilon}}\left|\frac{\partial u_{1,\varepsilon}}{\partial y}(x,y)\right| &\leq \sup_{K_{\varepsilon}}\left(\int_{x}^{f^{-1}(y)}|F(\xi,y,u_{\varepsilon}(\xi,y)|)d\xi\right)\\ &\leq \frac{2a}{\varepsilon}m(K_{\varepsilon},0)\\ &\leq \frac{2a}{\varepsilon}C(0)M_{\varepsilon},\end{aligned}$$

so: $\left(P_{K,(0,1)}(u_{1,\varepsilon})\right)_{\varepsilon} \in A_{+}$, $\left(P_{K,(0,1)}(u_{\varepsilon})\right)_{\varepsilon} \in A_{+}$.

Consequently: $\left(P_{K,1}(u_{\varepsilon})\right)_{\varepsilon} \in A_{+}$.

c) Induction.

Let us suppose that, for every $l \leq n$, we have: $\left(P_{K,l}(u_{\varepsilon})\right)_{\varepsilon} \in A_{+}$ and let us show that involves $\left(P_{K,n+1}(u_{\varepsilon})\right)_{\varepsilon} \in A_{+}$.

We use the notations from theorem 4.2.1.1.

c1) Let us show first that, every $n \in \mathbb{N}$,

$$\left(P_{1,n}(u_\varepsilon)\right)_\varepsilon \in A_+,\ \left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

As we have:

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int_{\varepsilon x}^{y} F(x,\eta,u_\varepsilon(x,\eta))d\eta,$$

we deduce that:

$$\frac{\partial^2 u_{1,\varepsilon}}{\partial x^2}(x,y) = -\varepsilon F(x,\varepsilon x,\varphi(x)) + \int_{\varepsilon x}^{y} \frac{\partial}{\partial x} F(x,\eta,u_\varepsilon(x,\eta))d\eta$$

and, by successive derivations, for $n \geq 1$:

$$\frac{\partial^{n+1} u_{1,\varepsilon}}{\partial x^{n+1}}(x,y) = -n\varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} F(x,\varepsilon x,\varphi(x)) + \int_{\varepsilon x}^{y} \frac{\partial^n}{\partial x^n} F(x,\eta,u_\varepsilon(x,\eta))d\eta.$$

We have:

$$\sup_{(x,y)\in K_\varepsilon} \left|\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y)\right| \leq \sup_{x\in\left[-\frac{a}{\varepsilon},\frac{a}{\varepsilon}\right]} n\varepsilon \left|\frac{\partial^{n-1}}{\partial x^{n-1}} F(x,\varepsilon x,\varphi(x))\right| + 2a \sup_{(x,y)\in K_\varepsilon} \left|\frac{\partial^n}{\partial x^n} F(x,y,u_\varepsilon(x,y))\right|,$$

now, from the property of stability:

$$\begin{aligned}\left(\sup_{(x,y)\in K_\varepsilon} \left|\frac{\partial^n}{\partial x^n} F(x,y,u_\varepsilon(x,y))\right|\right) &= P_{K_\varepsilon,(n,0)}(F(.,.,u_\varepsilon)) \leq P_{K_\varepsilon,n}(F(.,.,u_\varepsilon)) \\ &\leq \sum\nolimits_{i=0}^{i=n} C_i P^i_{K_\varepsilon,n}(u_\varepsilon)\end{aligned}$$

and:

$$\sup_{x\in\left[-\frac{a}{\varepsilon},\frac{a}{\varepsilon}\right]} n\varepsilon \left|\frac{\partial^{n-1}}{\partial x^{n-1}} F(x,\varepsilon x,\varphi(x))\right| \leq n\varepsilon \left(m\left(K_\varepsilon, n-1\right)\right) \leq n\varepsilon C\left(n-1\right) M_\varepsilon,$$

so:

$$\left(P_{K,(n+1,0)}\left(u_{1,\varepsilon}\right)\right)_\varepsilon \in A_+,$$

hence: $\left(P_{K,(n+1,0)}\left(u_\varepsilon\right)\right)_\varepsilon \in A_+$.

Let us show that, for every $n \in \mathbb{N}$, $\left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A_+$.

As we have:

$$\frac{\partial u_\varepsilon}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) - \int_x^{\frac{y}{\varepsilon}} F(\xi, y, u_\varepsilon(\xi,y))d\xi,$$

we follow that:

$$\frac{\partial^2 u_{1,\varepsilon}}{\partial y^2}(x,y) = -\frac{1}{\varepsilon}F(\frac{y}{\varepsilon}, y, \varphi(\frac{y}{\varepsilon})) - \int_x^{\frac{y}{\varepsilon}} \frac{\partial}{\partial y}F(\xi, y, u_\varepsilon(\xi,y))d\xi$$

and, by successive derivations, for $n \geq 1$:

$$\frac{\partial^{n+1} u_{1,\varepsilon}}{\partial y^{n+1}}(x,y) = -n\frac{1}{\varepsilon}\frac{\partial^{n-1}}{\partial y^{n-1}}F(\frac{y}{\varepsilon}, y, \varphi(\frac{y}{\varepsilon})) - \int_x^{\frac{y}{\varepsilon}} \frac{\partial^n}{\partial y^n}F(\xi, y, u_\varepsilon(\xi,y))d\xi.$$

We have:

$$\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^{n+1} u_{1,\varepsilon}}{\partial y^{n+1}}(x,y)\right| \leq \sup_{y\in[-a,a]} n\frac{1}{\varepsilon}\left|\frac{\partial^{n-1}}{\partial y^{n-1}}F(\frac{y}{\varepsilon}, y, \varphi(\frac{y}{\varepsilon}))\right| + 2\lambda \sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|,$$

now, from the property of stability:

$$\begin{aligned}\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right| &= P_{K,(0,n)}(F(.,.,u_\varepsilon)) \\ &\leq P_{K,n}(F(.,.,u_\varepsilon)) \\ &\leq \sum\nolimits_{i=0}^{i=n} C_i P^i_{K_\varepsilon,n}(u_\varepsilon)\end{aligned}$$

and:

$$\sup_{y\in[-a,a]} n\frac{1}{\varepsilon}\left|\frac{\partial^{n-1}}{\partial y^{n-1}}F(\frac{y}{\varepsilon}, y, \varphi(\frac{y}{\varepsilon}))\right| \leq n\frac{1}{\varepsilon}\left(m\left(K_\varepsilon, n-1\right)\right) \leq n\frac{1}{\varepsilon}C\left(n-1\right)M_\varepsilon,$$

so, for every $K \Subset \mathbb{R}^2$ and for every $n \in \mathbb{N}$,

$$\left(P_{K,(0,n+1)}\left(u_{1,\varepsilon}\right)\right)_\varepsilon \in A_+,$$

hence: $\left(P_{K,(0,n+1)}\left(u_\varepsilon\right)\right)_\varepsilon \in A_+$.

c2) For $\alpha + \beta = n$ and $\beta \geq 1$, we have now:

$$\begin{aligned} P_{K,(\alpha+1,\beta)}(u_\varepsilon) &= \sup_{(x,y)\in K} \left| D^{(\alpha+1,\beta)} u_\varepsilon(x,y) \right| = \sup_{(x,y)\in K} \left| D^{(\alpha,\beta-1)} D^{(1,1)} u_\varepsilon(x,y) \right| \\ &= \sup_{(x,y)\in K} \left| D^{(\alpha,\beta-1)} F(x,y,u_\varepsilon(x,y)) \right| = P_{K,(\alpha,\beta-1)}(F(.,.,u_\varepsilon)) \\ &\leq P_{K,n-1}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)), \end{aligned}$$

so we have:

$$P_{3,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_\varepsilon) \leq P_{K,n}(F(.,.,u_\varepsilon))$$

and then, the hypothesis of stability assures that:

$$\left(P_{3,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

In the same way, for $\alpha + \beta = n$ and $\alpha \geq 1$, we have:

$$\begin{aligned} P_{K,(\alpha,\beta+1)}(u_\varepsilon) &= \sup_{(x,y)\in K} \left| D^{(\alpha,\beta+1)} u_\varepsilon(x,y) \right| = \sup_{(x,y)\in K} \left| D^{(\alpha-1,\beta)} D^{(1,1)} u_\varepsilon(x,y) \right| \\ &= \sup_{(x,y)\in K} \left| D^{(\alpha-1,\beta)} F(x,y,u_\varepsilon(x,y)) \right| = P_{K,(\alpha-1,\beta)}(F(.,.,u_\varepsilon)) \\ &\leq P_{K,n-1}(F(.,.,u_\varepsilon)) \leq P_{K,n}(F(.,.,u_\varepsilon)). \end{aligned}$$

So we finally have:

$$P_{4,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\varepsilon) \leq P_{K,n}(F(.,.,u_\varepsilon))$$

and the hypothesis of stability assures that:

$$\left(P_{4,n}(u_\varepsilon)\right)_\varepsilon \in A_+.$$

In conclusion, we actually have:

$$\left(P_{K,n+1}(u_\varepsilon)\right)_\epsilon \in A_+.$$

□

8.2.1.3. Consequence

So $u = [u_\epsilon]$ *is a generalized function which we can consider as the generalized solution to the characteristic Cauchy problem* (P_C).

8.2.1.4. Open question

How this generalized function depends on the approximation of $\{y = 0\}$ *by* $\{y = \varepsilon x\}$ *?*

The question remains open.

8.3 Case of irregular data

8.3.1.1.Notations, reminders and hypothesis

We can also give a meaning to the characteristic Cauchy problem (P_C) in the case where φ and ψ are themselves irregular data (for example some generalized functions) by beginning to solve:

$$P_{(\varepsilon,\eta)}\begin{cases} \dfrac{\partial^2 u_{(\varepsilon,\eta)}}{\partial x \partial y}(x,y) = F(x,y,u_{(\varepsilon,\eta)}(x,y)) \\ u_{(\varepsilon,\eta)}\,(x,\varepsilon x) = \varphi_\eta(x) \\ \dfrac{\partial u_{(\varepsilon,\eta)}}{\partial y}\,(x,\varepsilon x) = \psi_\eta(x), \end{cases}$$

where $(\varphi_\eta)_\eta$ and $(\psi_\eta)_\eta$ are representatives of φ and ψ in an appropriate algebra.

The parameter ε permits to replace the given problem by a non-characteristic one, whereas the parameter η makes it regular.

$$u_{0,(\varepsilon,\eta)}(x,y) = \varphi_\eta(x) - \varepsilon\Psi_\eta\,(x) + \varepsilon\Psi\eta\left(\frac{y}{\varepsilon}\right);$$

$$u_{(\varepsilon,\eta)}(x,y) = u_{0,(\varepsilon,\eta)}(x,y) - \iint_{D_\varepsilon(x,y)} F(\xi,\theta,u_{(\varepsilon,\eta)}(\xi,\theta))d\xi d\theta.$$

Here we make the following hypothesis:

Keeping hypothesis $(H1)$ from previous theorem.

Moreover we suppose that:

$$(H5)\begin{cases} \exists\,\left(r_{\varepsilon,\eta}\right)_{(\varepsilon,\eta)}\in\mathbb{R}_*^{]0,1]\times]0,1]} \text{ such that } \forall K_2\Subset\mathbb{R}, \forall\alpha_2\in\mathbb{N},\exists D_2\in\mathbb{R}_+^*,\exists p\in\mathbb{N},\\ \max\left[\sup\limits_{K_2}\left|D^{\alpha_2}\varphi_\eta(\frac{y}{\varepsilon})\right|,\sup\limits_{K_2}\left|D^{\alpha_2}\Psi_\eta(\frac{y}{\varepsilon})\right|\right]\le\frac{D_2}{(r_{\varepsilon,\eta})^p}\end{cases}$$

$$(H_6)\begin{cases} \mathcal{C}=A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{]0,1]\times]0,1]}\text{:}\\ (\varepsilon)_{(\varepsilon,\eta)}\,;\left(r_{\varepsilon,\eta}\right)_{(\varepsilon,\eta)}\,;\left(e^{\frac{m_\varepsilon}{\varepsilon}}\right)_{(\varepsilon,\eta)}\,;\left(M_\varepsilon\right)_{(\varepsilon,\eta)}.\end{cases}$$

$$(H_7)\begin{cases} \mathcal{A}\left(\mathbb{R}^2\right)=\mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2) \text{ is built on } \mathcal{C}\\ \text{with } (\mathcal{E},\mathcal{P})=\left(\mathrm{C}^\infty(\mathbb{R}^2),(P_{K,l})_{K\Subset\mathbb{R}^2,l\in\mathbb{N}}\right)\\ \text{and } \mathcal{A}\left(\mathbb{R}^2\right) \text{ is stable under } F \text{ relativly to } \mathcal{C}.\end{cases}$$

8.3.1.2. Theorem

With the notations and the hypothesis of the above paragraph 8.3.1.1., if $u_{(\varepsilon,\eta)}$ *is the solution to the problem* $P_{(\varepsilon,\eta)}$*, therefore the family* $(u_{(\varepsilon,\eta)})_{(\varepsilon,\eta)}$ *is the representative of a generalized function which belongs to algebra* $\mathcal{A}(\mathbb{R}^2)$.

Proof.

We have:

$$u_{(\varepsilon,\eta)}(x,y)=u_{0,(\varepsilon,\eta)}(x,y)-\iint_{D_\varepsilon(x,y)}F(\xi,\theta,u_{(\varepsilon,\eta)}(\xi,\theta))d\xi d\theta,$$

where:

$$u_{0,(\varepsilon,\eta)}(x,y)=\varphi_\eta(x)-\varepsilon\Psi_\eta\left(x\right)+\varepsilon\Psi\eta\left(\frac{y}{\varepsilon}\right),$$

Ψ being a primitive of ψ, and:

$$u_{1,(\varepsilon,\eta)}(x,y)=\iint_{D_\varepsilon(x,y)}F(\xi,\theta,u_{(\varepsilon,\eta)}(\xi,\theta))d\xi d\theta$$

a) For $K=K_1\times K_2=[-a;a]\times[-a;a]$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$, there exists $C_1>0$ and

$C_2 > 0$ such that:

$$\sup_{K_1} |D^{\alpha_1}\varphi_\eta(x)| \leq C_1 (K_1, \alpha_1);$$

$$\varepsilon \sup_{K_1} |D^{\alpha_1}\Psi_\eta(x)| \leq \varepsilon C_2 (K_1, \alpha 1).$$

$G_\eta (y) = \Psi_\eta \circ f_\varepsilon^{-1} (y) = \Psi_\eta \left(\frac{y}{\varepsilon}\right)$, we can write:

$$\varepsilon \sup_{K_2} |D^{\alpha_2} G_\eta (y)| \leq \frac{D_2}{\varepsilon^{\alpha_2 - 1} (r_{\varepsilon,\eta})^{p(\alpha_2, K_2)}},$$

so $\left(P_{K,\alpha}\left(u_{0,(\varepsilon,\eta)}\right)\right)_{(\varepsilon,\eta)} \in A_+$.

b) We have to show that, for every integer n: $\left(P_{K,n}\left(u_{(\varepsilon,\eta)}\right)\right)_{(\varepsilon,\eta)} \in A_+$.

We have:

$$u_{1,(\varepsilon,\eta)}(x,y) = \iint_{D_\varepsilon(x,y)} F(\xi, \theta, u_{(\varepsilon,\eta)}(\xi,\theta)) d\xi d\theta.$$

According to the above results:

$$\sup_K \left| \iint_{D_\varepsilon(x,y)} F(\xi, \theta, u_{(\varepsilon,\eta)}(\xi,\theta)) d\xi d\theta \right| \leq \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda m_\lambda (f(\lambda) - f(-\lambda))],$$

with: $f(x) = \varepsilon x$; $\lambda = \dfrac{a}{\varepsilon}$; $m_\lambda = m_\varepsilon$; so $(f(\lambda) - f(-\lambda)) = 2a$ and

$$2\lambda m_\lambda (f(\lambda) - f(-\lambda)) = 2\frac{a}{\varepsilon} 2a m_\varepsilon = 4\frac{a^2}{\varepsilon} m_\varepsilon,$$

where:

$$\sup_{K\varepsilon} \left| u_{1,(\varepsilon,\eta)}(x,y) \right| \leq \frac{\Phi_\varepsilon}{m_\varepsilon} e^{\frac{4a^2}{\varepsilon} m_\varepsilon},$$

with:

$$\Phi_\varepsilon = \sup_{K\varepsilon} |F(x,y,0)| + m_\varepsilon \left\| u_{0,(\varepsilon,\eta)} \right\|_{\infty, K_\varepsilon} \leq C(0) M_\varepsilon + m_\varepsilon \left(\frac{3D_2}{(r_{\varepsilon,\eta})^{p1}} \right),$$

where: $p1 = p([-a,a], 0)$.

So $\left(P_{K,0}\left(u_{1,(\varepsilon,\eta)}\right)\right)_{(\varepsilon,\eta)} \in A_+$, hence: $\left(P_{K,0}\left(u_{(\varepsilon,\eta)}\right)\right)_{(\varepsilon,\eta)} \in A_+$.

Moreover:

$$\frac{\partial u_{(\varepsilon,\eta)}}{\partial x}(x,y) = \frac{\partial u_{0,(\varepsilon,\eta)}}{\partial x}(x,y) + \int_{f(x)}^{y} F(x, \theta, u_{(\varepsilon,\eta)}(x,\theta)) d\theta.$$

We have:

$$\frac{\partial u_{1,(\varepsilon,\eta)}}{\partial x} = \int_{f(x)}^{y} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))d\theta,$$

so, according to hypothesis (H_1):

$$\sup_{K_\varepsilon}\left(\int_{f(x)}^{y} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))d\theta\right) \leq 2am(K_\varepsilon,0),$$

then: $\big(P_{K,(1,0)}(u_{1,(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+$, consequently: $\big(P_{K,(1,0)}(u_{(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+$.

We have:

$$\frac{\partial u_{(\varepsilon,\eta)}}{\partial y}(x,y) = \frac{\partial u_{0,(\varepsilon,\eta)}}{\partial y}(x,y) - \int_{x}^{f^{-1}(y)} F(\xi,y,u_{(\varepsilon,\eta)}(\xi,y))d\xi.$$

In the same way, we get:

$$\begin{aligned}\sup_{K_\varepsilon}\left|\frac{\partial u_{1,(\varepsilon,\eta)}}{\partial y}(x,y)\right| &\leq \sup_{K_\varepsilon}\left(\int_{x}^{f^{-1}(y)} \big|F(\xi,y,u_{(\varepsilon,\eta)}(\xi,y)\big|)d\xi\right)\\ &\leq \frac{2a}{\varepsilon}m(K_\varepsilon,0)\\ &\leq \frac{2a}{\varepsilon}C(0)M_\varepsilon,\end{aligned}$$

so: $\big(P_{K,(0,1)}(u_{1,(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+$, $\big(P_{K,(0,1)}(u_{(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+$.

Consequently: $\big(P_{K,1}(u_{(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+$.

c) Induction.

Let us suppose that, for every $l \leq n$, we have: $\big(P_{K,l}(u_{(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+$ and let us show that involves $(P_{K,n+1}(u_{(\varepsilon,\eta)})_{(\varepsilon,\eta)} \in A_+$.

we use the notations from theorem 4.2.1.1.

c1) Let us show first that, for every $n \in \mathbb{N}$,

$$\big(P_{1,n}(u_{(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+,\ \big(P_{2,n}(u_{(\varepsilon,\eta)})\big)_{(\varepsilon,\eta)} \in A_+.$$

As we have:

$$\frac{\partial u_{(\varepsilon,\eta)}}{\partial x}(x,y) = \frac{\partial u_{0,(\varepsilon,\eta)}}{\partial x}(x,y) + \int_{\varepsilon x}^{y} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))d\theta,$$

we deduce that:

$$\frac{\partial^2 u_{1,(\varepsilon,\eta)}}{\partial x^2}(x,y) = -\varepsilon F(x,\varepsilon x,\varphi_\eta(x)) + \int_{\varepsilon x}^{y} \frac{\partial}{\partial x} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))d\theta$$

and, by successive derivations, for $n \geq 1$:

$$\frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial x^{n+1}}(x,y) = -n\varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} F(x,\varepsilon x,\varphi_\eta(x)) + \int_{\varepsilon x}^{y} \frac{\partial^n}{\partial x^n} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))d\theta.$$

We have:

$$\sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial x^{n+1}}(x,y) \right| \leq \sup_{x\in\left[-\frac{a}{\varepsilon},\frac{a}{\varepsilon}\right]} n\varepsilon \left| \frac{\partial^{n-1}}{\partial x^{n-1}} F(x,\varepsilon x,\varphi_\eta(x)) \right| + 2a \sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_{(\varepsilon,\eta)}(x,y)) \right|,$$

now, acording to the property of stability:

$$\begin{aligned} \left(\sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_{(\varepsilon,\eta)}(x,y)) \right| \right) &= P_{K_\varepsilon,(n,0)}(F(.,.,u_{(\varepsilon,\eta)})) \leq P_{K_\varepsilon,n}(F(.,.,u_{(\varepsilon,\eta)})) \\ &\leq \sum\nolimits_{i=0}^{i=n} C_i P^i_{K_\varepsilon,n}(u_{(\varepsilon,\eta)}), \end{aligned}$$

and:

$$\sup_{x\in\left[-\frac{a}{\varepsilon},\frac{a}{\varepsilon}\right]} n\varepsilon \left| \frac{\partial^{n-1}}{\partial x^{n-1}} F(x,\varepsilon x,\varphi_\varepsilon(x)) \right| \leq n\varepsilon \left(m\left(K_\varepsilon, n-1\right)\right) \leq n\varepsilon C\left(n-1\right) M_\varepsilon,$$

so:

$$\left(P_{K,(n+1,0)} \left(u_{1,(\varepsilon,\eta)} \right) \right)_{(\varepsilon,\eta)} \in A_+,$$

hence: $\left(P_{K,(n+1,0)} \left(u_{(\varepsilon,\eta)} \right) \right)_{(\varepsilon,\eta)} \in A_+$.

Let us show that, for every $n \in \mathbb{N}$, $\left(P_{2,n}(u_{(\varepsilon,\eta)}) \right)_{(\varepsilon,\eta)} \in A_+$.

As we have:

$$\frac{\partial u_{(\varepsilon,\eta)}}{\partial y}(x,y) = \frac{\partial u_{0,(\varepsilon,\eta)}}{\partial y}(x,y) - \int_{x}^{\frac{y}{\varepsilon}} F(\xi,y,u_{(\varepsilon,\eta)}(\xi,y))d\xi,$$

we deduce that:

$$\frac{\partial^2 u_{1,(\varepsilon,\eta)}}{\partial y^2}(x,y) = -\frac{1}{\varepsilon}F(\frac{y}{\varepsilon},y,\varphi_\eta(\frac{y}{\varepsilon})) - \int_x^{\frac{y}{\varepsilon}} \frac{\partial}{\partial y}F(\xi,y,u_{(\varepsilon,\eta)}(\xi,y))d\xi$$

and, by successive derivations, for $n \geq 1$:

$$\frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial y^{n+1}}(x,y) = -n\frac{1}{\varepsilon}\frac{\partial^{n-1}}{\partial y^{n-1}}F(\frac{y}{\varepsilon},y,\varphi_\eta(\frac{y}{\varepsilon})) - \int_x^{\frac{y}{\varepsilon}} \frac{\partial^n}{\partial y^n}F(\xi,y,u_{(\varepsilon,\eta)}(\xi,y))d\xi.$$

We have:

$$\begin{aligned}&\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial y^{n+1}}(x,y)\right|\\ &\quad\leq \sup_{y\in[-a,a]} n\frac{1}{\varepsilon}\left|\frac{\partial^{n-1}}{\partial y^{n-1}}F(\frac{y}{\varepsilon},y,\varphi_\eta(\frac{y}{\varepsilon}))\right| + 2\lambda \sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_{(\varepsilon,\eta)}(x,y))\right|,\end{aligned}$$

now, according to the property of stability:

$$\begin{aligned}\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_{(\varepsilon,\eta)}(x,y))\right| &= P_{K,(0,n)}(F(.,.,u_{(\varepsilon,\eta)}))\\ &\leq P_{K,n}(F(.,.,u_{(\varepsilon,\eta)}))\\ &\leq \sum\nolimits_{i=0}^{i=n} C_i P^i_{K_\varepsilon,n}(u_{(\varepsilon,\eta)})\end{aligned}$$

and:

$$\sup_{y\in[-a,a]} n\frac{1}{\varepsilon}\left|\frac{\partial^{n-1}}{\partial y^{n-1}}F(\frac{y}{\varepsilon},y,\varphi_\eta(\frac{y}{\varepsilon}))\right| \leq n\frac{1}{\varepsilon}m\left(K_\varepsilon,n-1\right) \leq n\frac{1}{\varepsilon}C\left(n-1\right)M_\varepsilon,$$

so, for every $K \Subset \mathbb{R}^2$ and every $n \in \mathbb{N}$,

$$\left(P_{K,(0,n+1)}\left(u_{1,(\varepsilon,\eta)}\right)\right)_{(\varepsilon,\eta)} \in A_+,$$

hence: $\left(P_{K,(0,n+1)}\left(u_{(\varepsilon,\eta)}\right)\right)_{(\varepsilon,\eta)} \in A_+$.

c2) For $\alpha+\beta=n$ and $\beta \geq 1$, we have now:

$$\begin{aligned}P_{K,(\alpha+1,\beta)}(u_{(\varepsilon,\eta)}) &= \sup_{(x,y)\in K}\left|D^{(\alpha+1,\beta)}u_{(\varepsilon,\eta)}\left(x,y\right)\right| = \sup_{(x,y)\in K}\left|D^{(\alpha,\beta-1)}D^{(1,1)}u_{(\varepsilon,\eta)}\left(x,y\right)\right|\\ &= \sup_{(x,y)\in K}\left|D^{(\alpha,\beta-1)}F(x,y,u_{(\varepsilon,\eta)}\left(x,y\right))\right| = P_{K,(\alpha,\beta-1)}(F(.,.,u_{(\varepsilon,\eta)}))\\ &\leq P_{K,n-1}(F(.,.,u_{(\varepsilon,\eta)})) \leq P_{K,n}(F(.,.,u_{(\varepsilon,\eta)})).\end{aligned}$$

So we finally have:

$$P_{3,n}(u_{(\varepsilon,\eta)}) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_{(\varepsilon,\eta)}) \leq P_{K,n}(F(.,.,u_{(\varepsilon,\eta)}))$$

and the hypothesis of stability assures that:

$$\left(P_{3,n}(u_{(\varepsilon,\eta)})\right)_{(\varepsilon,\eta)} \in A_+.$$

In the same way, for $\alpha+\beta=n$ and $\alpha\geq 1$, we have:

$$\begin{aligned} P_{K,(\alpha,\beta+1)}(u_{(\varepsilon,\eta)}) &= \sup_{(x,y)\in K} \left| D^{(\alpha,\beta+1)} u_{(\varepsilon,\eta)}\,(x,y) \right| = \sup_{(x,y)\in K} \left| D^{(\alpha-1,\beta)} D^{(1,1)} u_{(\varepsilon,\eta)}\,(x,y) \right| \\ &= \sup_{(x,y)\in K} \left| D^{(\alpha-1,\beta)} F(x,y,u_{(\varepsilon,\eta)}\,(x,y)) \right| = P_{K,(\alpha-1,\beta)}(F(.,.,u_{(\varepsilon,\eta)})) \\ &\leq P_{K,n-1}(F(.,.,u_{(\varepsilon,\eta)})) \leq P_{K,n}(F(.,.,u_{(\varepsilon,\eta)})). \end{aligned}$$

So we have:

$$P_{4,n}(u_{(\varepsilon,\eta)}) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_{(\varepsilon,\eta)}) \leq P_{K,n}(F(.,.,u_{(\varepsilon,\eta)}))$$

and the hypothesis of stability assures that:

$$\left(P_{4,n}(u_{(\varepsilon,\eta)})\right)_{(\varepsilon,\eta)} \in A_+.$$

In conclusion, we have indeed:

$$\left(P_{K,n+1}(u_{(\varepsilon,\eta)})\right)_{(\varepsilon,\eta)} \in A_+.$$

□

8.3.1.3. Consequence

So $u = \left[u_{(\varepsilon,\eta)}\right]$ *is a generalized function we can consider as the generalized solution to the characteristic Cauchy problem* P_C. □

8.4 Qualitative study of the solution. Case $F = 0$

$\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under F, $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}\left(\mathbb{R}^2\right)$ are built on the same ring of generalized constants as before.

Generalized solution to a characteristic Cauchy problem

8.4.1.1. Proposition

We consider the characteristic Cauchy problem:

$$(P_C)\begin{cases} \dfrac{\partial^2}{\partial x \partial y}u = 0 \\ u_{|(Ox)} = \varphi \\ \dfrac{\partial u}{\partial y} \;|_{(Ox)} = \psi. \end{cases}$$

We suppose that φ *and* ψ *are smooth and that* Ψ *verifies:*

$$\Psi \in \mathcal{L}^1(\mathbb{R}),\ \int_{\mathbb{R}} \Psi(t)dt = 1,$$

$$\forall K \Subset \mathbb{R}, \forall \alpha \in \mathbb{N}, \exists C > 0, p \in \mathbb{N} : \sup_{y \in K} \left| \Psi^{(\alpha)}\left(\frac{y}{\varepsilon}\right) \right| \leq \frac{C}{\varepsilon^p}.$$

Let δ_x *(respectively* δ_y*) the Dirac distribution which acts on the functions of variable* x *(resp* y*),* 1_x *(resp* 1_y*) the constant function equal to 1 of variable* x *(resp* y*).*

$$u_\varepsilon(x, y) = \varphi(x) - \varepsilon\Psi(x) + \varepsilon\Psi\left(\frac{y}{\varepsilon}\right) = \varphi(x) - \varepsilon\Psi(x) + \varepsilon^2\left(\frac{1}{\varepsilon}\Psi\left(\frac{y}{\varepsilon}\right)\right),$$

then: $[u_\varepsilon]$ *is the generalized solution to the characteristic irregular Cauchy problem.*

$$[u_\varepsilon] = u_1 + [\varepsilon u_2] + \left[\varepsilon^2 u_{\varepsilon,3}\right],$$

with:

$$\begin{cases} u_1 = 1_y \otimes \varphi \in \mathrm{C}^\infty(\mathbb{R}^2) \\ u_2 = -1_y \otimes \Psi \in \mathrm{C}^\infty(\mathbb{R}^2) \\ [u_{\varepsilon,3}] \sim 1_x \otimes \delta_y \in \mathcal{D}'(\mathbb{R}^2). \end{cases}$$

Proof.

$u_\varepsilon = u_1 + \varepsilon u_2 + \varepsilon^2 u_{\varepsilon,3}$, with: $u_2(x,y) = -\Psi(x)$ and:

$$u_{\varepsilon,3}(X,Y) = \frac{1}{\varepsilon}\Psi\left(\frac{y}{\varepsilon}\right).$$

Then, according to the hypothesis:

$$\forall H \times K \Subset \mathbb{R}^2, \forall(\beta,\alpha) \in \mathbb{N}^2, \exists C_3 > 0, p \in \mathbb{N} : \sup_{H\times K}\left|\frac{\partial^{\beta+\alpha}}{(\partial x)^\beta(\partial y)^\alpha} u_{(\varepsilon,\eta),3}(x,y)\right| \leq \frac{C_3}{\varepsilon^{\alpha+1}\varepsilon^p},$$

where: $(u_{\varepsilon,3})_\varepsilon \in \mathcal{H}_{(A,\varepsilon,\mathcal{P})}(\mathbb{R}^2)$.

We have also: $(u_1 + \varepsilon u_2)_\varepsilon \in \mathcal{H}_{(A,\varepsilon,\mathcal{P})}(\mathbb{R}^2)$.

Moreover:

$$\left[\varepsilon^2 u_{\varepsilon,3}\right] = \left[\varepsilon^2.1_x \otimes \left(y \longmapsto \frac{1}{\varepsilon}\Psi\left(\frac{y}{\varepsilon}\right)\right)\right],$$

and: $\lim_{\substack{\mathcal{D}'(\mathbb{R}) \\ \varepsilon\longrightarrow 0}} \left(y \longmapsto \frac{1}{\varepsilon}\Psi\left(\frac{y}{\varepsilon}\right)\right) = \delta_y$. □

Case $\varphi = 0$, $\psi \sim \delta$

the association is defined as the following way:

Let us consider $g \in \mathcal{D}(\mathbb{R})$, verifying $\int_{\mathbb{R}} g(\xi)d\xi = 1$. Let us put: $\frac{1}{\eta}g(\frac{x}{\eta}) = \psi_\eta(x)$. Then $(\psi_\eta)_\eta$ have, in a distributional sense, δ as limit. So $\psi = [\psi_\eta]$ is actually associated to δ.

8.4.1.2. Proposition $(F = 0, \varphi = 0, \psi \sim \delta)$. [10]

The generalized solution u to the following characteristic irregular Cauchy problem:

$$(P_C)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = 0 \\ u_{|(Ox)} = 0 \\ \dfrac{\partial u}{\partial y} \,|_{(Ox)} = \delta. \end{cases}$$

is: $\left[u_{(\varepsilon,\eta)}\right] = \left[\varepsilon w_{(\varepsilon,\eta),1}\right] + \left[\varepsilon w_{(\varepsilon,\eta),2}\right]$, *with:*

$$\begin{cases} \left[w_{(\varepsilon,\eta),1}\right] \sim 1_x \otimes Y_y \\ \left[w_{(\varepsilon,\eta),2}\right] \sim -Y_x \otimes 1_y. \end{cases}$$

Proof.

By considering as data the curve γ_ε of equation $y = \varepsilon x$, we can solve the following non-characteristic problem:

$$(P_\varepsilon) \begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = 0 \\ u_{|\gamma_\varepsilon} = 0 \\ \dfrac{\partial u}{\partial y} \,|_{\gamma_\varepsilon} = \delta \end{cases} ;$$

and by putting the data regularized by the mollifiers ψ_η on the curve $\gamma_\varepsilon = \{y = \varepsilon x\}$, we can solve the non-characteristic regular problem:

$$\left(P_{(\varepsilon,\eta)}\right) \begin{cases} \dfrac{\partial^2 u_{(\varepsilon,\eta)}}{\partial x \partial y}(x,y) = 0 \\ u_{(\varepsilon,\eta)}\left(x, \varepsilon x\right) = 0 \\ \dfrac{\partial u_{(\varepsilon,\eta)}}{\partial y}\left(x, \varepsilon x\right) = \psi_\eta(x). \end{cases}$$

Let us determine solution u.

$$u_{(\varepsilon,\eta)}(x,y) = \int_0^y \psi_\eta\left(\frac{\theta}{\varepsilon}\right) d\theta - \int_0^{\varepsilon x} \psi_\eta\left(\frac{\theta}{\varepsilon}\right) d\theta - \iint_{D_\varepsilon(x,y)} F(\xi, \theta, u_{(\varepsilon,\eta)}(\xi,\theta)) d\xi d\theta$$

so: $u_{(\varepsilon,\eta)}(x,y) = \varepsilon \Psi_\eta\left(\frac{y}{\varepsilon}\right) - \varepsilon \Psi_\eta(x)$ where $\Psi_\eta(x) = \displaystyle\int_0^x \psi_\eta(t) dt$.

Hence: $u_{(\varepsilon,\eta)}(x,y) = \varepsilon w_{(\varepsilon,\eta),1} + \varepsilon w_{(\varepsilon,\eta),2}$.

$$\lim_{\eta \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} 0} \Psi_\eta = Y \text{ and } \lim_{(\varepsilon,\eta) \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} (0,0)} \left(y \mapsto \Psi_\eta\left(\frac{y}{\varepsilon}\right)\right) = Y.$$

So:

$$\begin{cases} \left[w_{(\varepsilon,\eta),1}\right] \sim 1_x \otimes Y_y \\ \left[w_{(\varepsilon,\eta),2}\right] \sim -Y_x \otimes 1_y, \end{cases}$$

hence:

$$\left[u_{(\varepsilon,\eta)}\right] = \left[\varepsilon w_{(\varepsilon,\eta),1}\right] + \left[\varepsilon w_{(\varepsilon,\eta),2}\right].$$

□

Case: $\varphi \sim S$**,** $\Psi \sim T$**;** $S \in \mathcal{D}'(\mathbb{R})$**,** $T \in \mathcal{D}'(\mathbb{R})$

the association being achieved by:

$$\varphi = [g_\eta * S] \text{ and } \Psi = [g_\eta * T]$$

since:

$$\lim_{\eta \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} 0} (g_\eta * S)_\eta = S \text{ and } \lim_{\eta \underset{\mathcal{D}'(\mathbb{R})}{\rightarrow} 0} (g_\eta * T)_\eta = T.$$

Terms of the problem.

We search a generalised solution u to the following characteristic irregular Cauchy problem:

$$(P_C)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = 0 \\ u_{|(Ox)} = S \\ \dfrac{\partial u}{\partial y} \mid_{(Ox)} = T'. \end{cases}$$

By considering the curve γ_ε of equation $y = \varepsilon x$ as data, we can solve the following non-characteristic problem:

$$(P_\varepsilon)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = 0 \\ u_{|\gamma_\varepsilon} = S \\ \dfrac{\partial u}{\partial y} \mid_{\gamma_\varepsilon} = T'; \end{cases}$$

and by putting the data regularized by mollifiers g_η on the curve $\gamma_\varepsilon = \{y = \varepsilon x\}$, we can solve the non-characteristic problem:

$$\left(P_{(\varepsilon,\eta)}\right)\begin{cases}\dfrac{\partial^2 u_{(\varepsilon,\eta)}}{\partial x\partial y}(x,y)=0\\ u_{(\varepsilon,\eta)}\left(x,\varepsilon x\right)=\left(g_\eta * S\right)(x)\\ \dfrac{\partial u_{(\varepsilon,\eta)}}{\partial y}\left(x,\varepsilon x\right)=\left(g_\eta * T'\right)(x).\end{cases}$$

Solving the problem.

For $g\in\mathcal{D}(\mathbb{R})$, with: $suppg=[-1;1]$, $0\le g\le 1$, $g(0)=1$, and $g^{(k)}(0)=0$ for every $k\in\mathbb{N}^*$; let us consider, for $x\in\mathbb{R}$

$$g_\eta(x)=\frac{1}{\eta}g(\frac{x}{\eta}).$$

Then $\left(g_\eta\right)_\eta$ converges in a distributional sense, to δ_x.

Let us determine the solution $u_{(\varepsilon,\eta)}$ to $\left(P_{(\varepsilon,\eta)}\right)$.

We have:

$$\begin{aligned}u_{(\varepsilon,\eta)}(x,y)&=\varepsilon\Psi_\eta(\frac{y}{\varepsilon})-\varepsilon\Psi_\eta(x)+\varphi_\eta(x)\\&=\varepsilon\left(g_\eta*T\right)(\frac{y}{\varepsilon})-\varepsilon\left(g_\eta*T\right)(x)+\left(g_\eta*S\right)(x).\end{aligned}$$

Hence:

$$\left[u_{(\varepsilon,\eta)}\right]=\left[\varepsilon u_{(\varepsilon,\eta),1}\right]+\left[\varepsilon u_{(\varepsilon,\eta),2}\right]+\left[u_{(\varepsilon,\eta),3}\right],$$

with:

$$\begin{cases}u_{(\varepsilon,\eta),1}\left(x,y\right)=\left(g_\eta*T\right)\left(\frac{y}{\varepsilon}\right)\\ \left[u_{(\varepsilon,\eta),2}\right]\sim -T_x\otimes 1_y\\ \left[u_{(\varepsilon,\eta),3}\right]\sim S_x\otimes 1_y\end{cases}$$

□

Case: $\varphi\sim\delta$**,** $\psi\sim\delta$

($\varphi\sim\delta$, $\psi\sim\delta$; $F=0$) so: ($\varphi\sim S\sim\delta$, $\Psi\sim Y\sim T$; $F=0$).

We search a generalized solution u to the following characteristic irregular Cauchy problem:

$$(P_C)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u = 0 \\ u_{|(Ox)} = \delta \\ \dfrac{\partial u}{\partial y} \,|_{(Ox)} = \delta. \end{cases}$$

From the above results:

$$u_{(\varepsilon,\eta)}(x,y) = \varepsilon\left(g_\eta * T\right)\left(\frac{y}{\varepsilon}\right) - \varepsilon\left(g_\eta * T\right)(x) + \left(g_\eta * S\right)(x).$$

Hence:

$$\left[u_{(\varepsilon,\eta)}\right] = \left[\varepsilon u_{(\varepsilon,\eta),1}\right] + \left[\varepsilon u_{(\varepsilon,\eta),2}\right] + \left[u_{(\varepsilon,\eta),3}\right],$$

with:

$$\begin{cases} \left[u_{(\varepsilon,\eta),1}\right] \sim 1_x \otimes Y_y \\ \left[u_{(\varepsilon,\eta),2}\right] \sim -Y_x \otimes 1_y \\ \left[u_{(\varepsilon,\eta),3}\right] \sim \delta_x \otimes 1_y. \end{cases}$$

Because if G is a primitive of g: $\lim\limits_{\eta \underset{\mathcal{D}'(\mathbb{R})}{\to} 0} G_\eta = Y$ and: $\lim\limits_{(\varepsilon,\eta) \underset{\mathcal{D}'(\mathbb{R})}{\to} (0,0)} \left(y \mapsto G_\eta(\frac{y}{\epsilon})\right) = Y$. $\square$

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Résumé de la thèse

Sur les singularités de certains problèmes différentiels.

Dans cette thèse nous proposons une méthode pour résoudre certains problèmes de Cauchy à données irrégulières ou caractéristiques en utilisant les récentes théories des fonctions généralisées. Nous étudions dans la première partie un problème de Cauchy et un problème de Goursat réguliers avec des données sur une courbe monotone. La deuxième partie est consacrée à la mise en place d'une algèbre adaptée à la résolution du problème de Cauchy généralisé. Dans la troisième partie nous donnons un sens à un problème de Cauchy généralisé et nous montrons qu'il admet une unique solution. Nous étudions de même un problème de Goursat généralisé. Dans la quatrième partie nous approchons un problème de Cauchy caractéristique par une famille de problèmes non caractéristiques (P_ε) en considérant la droite d'équation $y = \varepsilon x$. Si u_ε est la solution du problème (P_ε), $u = [u_\epsilon]$ est une fonction généralisée que nous considérons comme la solution généralisée du problème dans une algèbre convenablement définie. Nous donnons un sens au problème de Cauchy caractéristique dans le cas de données irrégulières en le remplaçant par une famille de problèmes non caractéristiques $\left(P_{(\varepsilon,\eta)}\right)$ dans une algèbre convenable. Le paramètre ε permet de se ramener à un problème non caractéristique que le paramètre η rend régulier. Si $u_{(\varepsilon,\eta)}$ est la solution du problème $\left(P_{(\varepsilon,\eta)}\right)$, $u = \left[u_{(\varepsilon,\eta)}\right]$ est une fonction généralisée que nous considérons comme la solution généralisée du problème.

Mots clés

Equations différentielles partielles non linéaires. Algèbre de fonctions généralisées. Problème de Cauchy caractéristique

Abstract

On the singularities of some differential problems.

In this thesis, we propose a method to solve some Cauchy problems with irregular or characteristic data by using the recent theories of generalized functions. We study a regular Cauchy problem and a regular Goursat problem in the first part with data on a monotonous curve. The second part is devoted to the setting up of an algebra adapted to the generalized Cauchy problem. In the third part, we give a meaning to a generalized Cauchy problem and we show that the problem admits a unique solution. We study a generalized Goursat problem in the same way. In the fourth part, we approach a characteristic Cauchy problem by a family of non-characteristic ones (P_ε) by considering the straight line of equation $y = \varepsilon x$. If u_ε is the solution to problem (P_ε), $u = [u_\epsilon]$ is a generalized function that we consider as the generalized solution to the problem in an appropriate algebra. We give a meaning to the characteristic Cauchy problem with irregular data by replacing it by a family of non-characteristic problems $\left(P_{(\varepsilon,\eta)}\right)$ in an appropriate algebra. The parameter ε permits to replace the given problem by a non-characteristic one, whereas the parameter η makes it regular. If $u_{(\varepsilon,\eta)}$ is the solution to problem $\left(P_{(\varepsilon,\eta)}\right)$, $u = \left[u_{(\varepsilon,\eta)}\right]$ is a generalized function considered as the generalized solution to the problem.

Keywords

Non linear Partial Differential Equations; Algebras of generalized functions; Characteristic Cauchy Problem.

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