Math-3150 Homework #5-Solutions, Summer 2005

3.8 #3 As $f_2 = g_1 = 0$ we have $B_n = C_n = 0$. Then we compute

$$\begin{aligned} A_n &= \frac{2}{2\sinh(n\pi/2)}\int_0^2 100\sin(n\pi x/2)dx \\ &= \frac{200}{\sinh(n\pi/2)}\left[\frac{-\cos(n\pi x/2)}{n\pi}\right]_0^2 \\ &= \frac{200}{\sinh(n\pi/2)}\frac{1-(-1)^n}{n\pi} \end{aligned}$$

We also compute

$$\begin{aligned} D_n &= \frac{2}{\sinh(2n\pi)}\int_0^1 100(1-y)\sin(n\pi y)dy \\ &= \frac{200}{\sinh(2n\pi)}\left[\frac{-\cos(n\pi y)}{n\pi} - \left(\frac{1}{n^2\pi^2}\sin(n\pi y) - \frac{y}{n\pi}\cos(n\pi y)\right)\right]_0^1 \\ &= \frac{200}{\sinh(2n\pi)}\left(\frac{1-(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi}\right) \\ &= \frac{200}{n\pi\sinh(2n\pi)} \end{aligned}$$

Then the solution is

$$\begin{aligned} u(x,y) = &\sum_{n=1}^{+\infty}\frac{200(1-(-1)^n)}{n\pi\sinh(n\pi/2)}\sin(n\pi x/2)\sinh(n\pi(1-y)/2) \\ &+\sum_{n=1}^{+\infty}\frac{200}{n\pi\sinh(2n\pi)}\sinh(n\pi x)\sin(n\pi y) \end{aligned}$$

3.8 #5 As all the functions are already Fourier sine series we have

$$\begin{cases} A_7 = 1/\sinh(7\pi) \text{ and all others are zero} \\ B_1 = 1/\sinh(\pi) \text{ and all others are zero} \\ C_3 = 1/\sinh(3\pi) \text{ and all others are zero} \\ D_6 = 1/\sinh(6\pi) \text{ and all others are zero} \end{cases}$$

Then the solution is

$$\begin{aligned} u(x,y) = &\frac{\sin(7\pi x)\sinh(7\pi(1-y))}{\sinh(7\pi)} + \frac{\sin(\pi x)\sinh(\pi y)}{\sinh\pi} \\ &+\frac{\sinh(3\pi(1-x))\sin(3\pi y)}{\sinh(3\pi)} + \frac{\sinh(6\pi x)\sin(6\pi y)}{\sinh(6\pi)} \end{aligned}$$

3.8 #11 a) We prove that $u(x,y,t) = w(x,y,t) + v(x,y)$ is a solution to the non-homogeneous heat problem by checking all conditions. First the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(w(x,y,t) + v(x,y)) = \frac{\partial w}{\partial t} = c^2\nabla^2 w$$

as w is a solution to the heat equation. Moreover we compute

$$\nabla^2 u = \nabla^2(w+v) = \nabla^2 w + \nabla^2 v = \nabla^2 w$$

then we have proved that

$$\frac{\partial u}{\partial t} = c^2\nabla^2 w$$

Then for the initial condition:

$$u(x,y,0) = w(x,y,0) + v(x,y) = f(x,y) - v(x,y) + v(x,y) = f(x,y)$$

as $w(x,y,0) = f(x,y) - v(x,y)$, so u satisfies also the initial condition. Now we check one boundary condition (the others are similar):

$$u(x,0,t) = w(x,0,t) + v(x,0) = f_1(x)$$

as $w(x,0,t) = 0$ (remeber that w is the solution to the homogeneous problem) and $v(x,0) = f_1(x)$. So we have proved that u is the solution to the non-homogeneous heat problem.

b) First we compute $v(x,y)$. As $f_1 = g_1 = g_2 = 0$ we have $A_n = C_n = D_n = 0$ and we compute

$$\begin{aligned} A_n &= \frac{2}{\sinh n\pi}\int_0^1 100\sin(n\pi x)dx \\ &= \frac{200}{\sinh(n\pi)}\left[\frac{-\cos(n\pi y)}{n\pi}\right]_0^1 \\ &= \frac{200(1-(-1)^n)}{n\pi\sinh(n\pi)} \end{aligned}$$

Then we have

$$v(x,y) = \sum_{n=1}^{+\infty}\frac{200(1-(-1)^n)}{n\pi\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi y)$$

Then we solve the homogeneous heat problem to find w. The initial condition is $\tilde{f}(x,y) = -v(x,y)$ so we compute

$$A_{mn} = 4\int_0^1\int_0^1 -\sum_{k=1}^{+\infty}\frac{200(1-(-1)^k)}{k\pi\sinh(k\pi)}\sin(k\pi x)\sinh(k\pi y)\sin m\pi x\sin n\pi y dxdy$$
$$= -4\sum_{k=1}^{+\infty}\frac{200(1-(-1)^k)}{k\pi\sinh(k\pi)}\int_0^1 \sin k\pi x\sin m\pi x dx\int_0^1 \sinh(k\pi y)\sin n\pi y dy$$

but as

$$\int_0^1 \sin k\pi x\sin m\pi x dx = \begin{cases} 0 \text{ if } k\neq m \\ 1/2 \text{ for } k=m \end{cases}$$

So we have (the sum over k disappears as only one term survives, the one where $k=m$):

$$A_{mn} = -4\frac{100(1-(-1)^m)}{m\pi\sinh(m\pi)}\int_0^1 \sinh(m\pi y)\sin n\pi y dy$$

Moreover you can prove (using the same technique as in the 1st midterm) that

$$\int_0^1 \sinh(m\pi y)\sin n\pi y dy = -\frac{n(-1)^n}{\pi(m^2+n^2)}\sinh(m\pi)$$

So we have

$$A_{mn} = -4\frac{100(1-(-1)^m)}{m\pi\sinh(m\pi)}\times\left(-\frac{n(-1)^n}{\pi(m^2+n^2)}\sinh(m\pi)\right) = \frac{400n(1-(-1)^m)(-1)^n}{\pi^2 m(m^2+n^2)}$$

and then the solution to the homogeneous problem is

$$w(x,y,t) = \sum_{m,n=1}^{+\infty}\frac{400n(1-(-1)^m)(-1)^n}{\pi^2 m(m^2+n^2)}e^{-\pi^2(m^2+n^2)t}\sin m\pi x\sin n\pi y$$

Then the solution to the non-homogeneous problem is

$$u(x,y,t) = w(x,y,t) + v(x,y)$$
$$= \sum_{m,n=1}^{+\infty}\frac{400n(1-(-1)^m)(-1)^n}{\pi^2 m(m^2+n^2)}e^{-\pi^2(m^2+n^2)t}\sin m\pi x\sin n\pi y$$
$$+\sum_{n=1}^{+\infty}\frac{200(1-(-1)^n)}{n\pi\sinh(n\pi)}\sin(n\pi x)\sinh(n\pi y)$$

4.2 #3 Wave equation in polar coordinates.
All A_n's are zero as $f(r) = 0$. We calculate

$$\begin{aligned} B_n &= \frac{2}{\alpha_n J_1^2(\alpha_n)} \int_0^1 g(r) J_0(\alpha_n r) r dr \\ &= \frac{2}{\alpha_n J_1^2(\alpha_n)} \int_0^{1/2} J_0(\alpha_n r) r dr \\ &= \frac{2}{\alpha_n J_1^2(\alpha_n)} \frac{(1/2)^2}{\alpha_n/2} J_1(\alpha_n/2) \end{aligned}$$

The last line is obtained by using the formula (11) page 202 with $a = 1/2$, $p = 0$ and $\alpha_n/2$. Then we find that

$$B_n = \frac{J_1(\alpha_n/2)}{\alpha_n^2 J_1^2(\alpha_n)}$$

Then the solution is

$$u(r,t) = \sum_{n=1}^{+\infty} \frac{J_1(\alpha_n/2)}{\alpha_n^2 J_1^2(\alpha_n)} \sin \alpha_n t J_0(\alpha_n r)$$

4.2 #5 Wave equation in polar coordinates.
Solve the wave equation in the unit disk ($a = 1$), with $c = 1$. The initial conditions are given by

$$f(r) = J_0(\alpha_1 r) \text{ and } g(r) = 0$$

As $f(r)$ is already a Fourier-Bessel series we have

$$A_1 = 1 \text{ and all other } A_n\text{'s are zero}$$

And all B_n's are zero as $g(r) = 0$. Then the solution is

$$u(r,t) = \cos(\alpha_1 t) J_0(\alpha_1 r)$$

4.2 #7 As $f(r)$ is a already a Bessel series we have

$$A_3 = 1 \text{ and all others are zero}$$

Next we compute the B_n's using formula (15) page 211 with $a = 1$ $k = 0$ and $\alpha = \alpha_n$:

$$\begin{aligned} B_n &= \frac{2}{\alpha_n J_1^2(\alpha_n)} \int_0^1 (1 - r^2) J_0(\alpha_n) r dr \\ &= \frac{2}{\alpha_n J_1^2(\alpha_n)} \times 2 \times \frac{1}{\alpha_n^2} J_2(\alpha_n) \\ &= \frac{4 J_2(\alpha_n)}{\alpha_n^3 J_1^2(\alpha_n)} \end{aligned}$$

Then the solution is

$$u(r,t) = J_0(\alpha_3 r) \cos \alpha_3 t + \sum_{n=1}^{+\infty} \frac{4 J_2(\alpha_n)}{\alpha_n^3 J_1^2(\alpha_n)} J_0(\alpha_n r) \sin \alpha_n t$$