Math-3150 Homework #4-Solutions, Summer 2005

3.5 #4 We compute:

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{200}{\pi}\int_0^{\pi/2} \sin nx dx$$
$$= \frac{200}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi/2} = \frac{200}{n\pi}\left(1-\cos(n\pi/2)\right)$$

So the solution is:

$$u(x,t) = \sum_{n=1}^{+\infty} \frac{200}{n\pi}\left(1-\cos(n\pi/2)\right)e^{-n^2 t}\sin nx$$

3.5 #14 We compute first the steady state $v(x) = Ax + B$:

$$\begin{cases} v(0) = 100 \\ v(\pi) = 50 \end{cases} \iff \begin{cases} B = 100 \\ A\pi = 50 \end{cases} \iff \begin{cases} B = 100 \\ A = 50/\pi \end{cases}$$

then $v(x) = \frac{50}{\pi}x + 100$.
Then now we solve the homogeneous heat problem with initial condition $f_1(x) = f(x) - v(x)$. We have already computed the coef of the Fourier Sine series of f in Ex 4 so we just have to compute those of v:

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{50}{\pi}x + 100\right)\sin nx dx = \frac{200}{\pi}\int_0^{\pi}\sin nx dx + \frac{100}{\pi^2}\int_0^{\pi} x\sin nx dx$$
$$= \frac{200}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} + \frac{100}{\pi^2}\left[\frac{1}{n^2}\sin nx - \frac{x}{n}\cos nx\right]_0^{\pi}$$
$$= \frac{200}{n\pi}\left(1-(-1)^n\right) - \frac{100}{n\pi}(-1)^n$$

Then we finally compute

$$b_n = \frac{200}{n\pi}\left(1-\cos(n\pi/2)\right) - \left(\frac{200}{n\pi}\left(1-(-1)^n\right) - \frac{100}{n\pi}(-1)^n\right)$$
$$= \frac{100}{n\pi}\left(-2\cos(n\pi/2) + 3(-1)^n\right)$$

So the solution to the homogeneous problem is:

$$u_1(x,t) = \sum_{n=1}^{+\infty} \frac{100}{n\pi}\left(-2\cos(n\pi/2) + 3(-1)^n\right)e^{-n^2 t}\sin nx$$

and then the solution to the non-homogeneous problem is:

$$u(x,t) = \sum_{n=1}^{+\infty} \frac{100}{n\pi}(-2\cos(n\pi/2) + 3(-1)^n)e^{-n^2 t}\sin nx + \frac{50}{\pi}x + 100$$

3.5 #15 The physical explanation is that as the initial condition is already the equilibrium state (the steady state) so the system should just stay still, that is the solution should just be the steady state for all time.
Let's see by computing the solution directly. First we have to build the associated homogeneous problem:

$$\begin{cases} \frac{\partial u}{\partial t} = c^2 \frac{\partial^2}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \ \forall t \\ u(x,0) = f(x) - u_1(x) = 0 \end{cases}$$

But we know the solution of this homogeneous problem: $w(x,t) = 0$ as the initial condition is the zero function which means all b_n's are zero. So finally the solution to the original problem is

$$u(x,t) = w(x,t) + u_1(x,t) = u_1(x,t)$$

as we predicted.

3.7 #3 The B_{mn} computation can be found in the book p 159:

$$\begin{aligned} B_{mn} &= 4\int_0^1\int_0^1 x(1-x)y(1-y)\sin(m\pi x)\sin(n\pi y)dxdy \\ &= 16\frac{((-1)^m - 1)((-1)^n - 1)}{\pi^6 m^3 n^3} \end{aligned}$$

We compute the $B^\star_{mn}$: $\lambda_{mn} = \sqrt{m^2+n^2}$ and g is already a double Fourier Sine series so we have

$$B^\star_{12} = \frac{1}{\sqrt{1^2+2^2}}2 = \frac{2}{\sqrt{5}} \text{ and all others are zero}$$

so the solution is

$$\begin{aligned} u(x,y,t) = \sum_{m,n=1}^{+\infty} & 16\frac{((-1)^m - 1)((-1)^n - 1)}{\pi^6 m^3 n^3}\cos(\sqrt{m^2+n^2}t)\sin(m\pi x)\sin(n\pi y) \\ & + \frac{2}{\sqrt{5}}\cos(\sqrt{5}t)\sin(\pi x)\sin(2\pi y) \end{aligned}$$

3.7 #12 The A_{mn} computation can be found in the book p 159:

$$A_{mn} = 4\int_0^1\int_0^1 x(1-x)y(1-y)\sin(m\pi x)\sin(n\pi y)dxdy$$
$$= 16\frac{((-1)^m-1)((-1)^n-1)}{\pi^6 m^3 n^3}$$

So the solution is

$$u(x,y,t) = \sum_{m,n=1}^{+\infty} 16\frac{((-1)^m-1)((-1)^n-1)}{\pi^6 m^3 n^3} e^{-\pi^2(m^2+n^2)t}\sin(m\pi x)\sin(n\pi y)$$

3.7 #13 As $f(x,y)$ is already a double Fourier Sine series we have

$$A_{11} = 1 \text{ and all others are zero}$$

So the solution is

$$u(x,y,t) = e^{-2\pi^2 t}\sin(\pi x)\sin(\pi y)$$

3.7 #14 We compute the A_{mn}:

$$A_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin(m\pi x)\sin(n\pi y)dxdy$$
$$= 4\iint_T \sin(m\pi x)\sin(n\pi y)dxdy$$

where T is the lower triangle in the unit square delimited by the line $y = x$

$$= 4\int_0^1\int_0^x \sin(m\pi x)\sin(n\pi y)dydx$$
$$= 4\int_0^1 \sin(m\pi x)\left[-\frac{\cos(n\pi y)}{n\pi}\right]_0^x dx$$
$$= \frac{-4}{n\pi}\int_0^1 \sin(m\pi x)(\cos(n\pi x)-1)dx$$
$$= \frac{-4}{n\pi}\left(-\int_0^1 \sin(m\pi x)dx + \int_0^1 \sin(m\pi x)\cos(n\pi x)dx\right)$$
$$= \frac{-4}{n\pi}\left(\left[\frac{\cos(m\pi x)}{m\pi}\right]_0^1 dx + \int_0^1 \sin(m\pi x)\cos(n\pi x)dx\right)$$
$$= \frac{-4}{n\pi}\left(\frac{(-1)^m-1}{m\pi} + \int_0^1 \sin(m\pi x)\cos(n\pi x)dx\right)$$

Then we have 2 cases. First for $m \neq n$ we have

$$\begin{aligned}
A_{mn} &= \frac{-4}{n\pi}\left(\frac{(-1)^m - 1}{m\pi} + \left[\frac{\cos((n-m)\pi x)}{2(n-m)\pi} - \frac{\cos((n+m)\pi x)}{2(n+m)\pi}\right]_0^1\right) \\
&= \frac{-4}{n\pi^2}\left(\frac{(-1)^m - 1}{m} + \left[\frac{\cos((n-m)\pi) - 1}{2(n-m)} - \frac{\cos((n+m)\pi) - 1}{2(n+m)}\right]\right) \\
&= \frac{-4}{n\pi^2}\left(\frac{(-1)^m - 1}{m} + \left[\frac{(-1)^{n-m} - 1}{2(n-m)} - \frac{(-1)^{n+m} - 1}{2(n+m)}\right]\right) \\
&= \frac{-4}{n\pi^2}\left(\frac{(-1)^m - 1}{m} + \frac{m((-1)^{n+m} - 1)}{n^2 - m^2}\right)
\end{aligned}$$

and for $m = n$ we have

$$\begin{aligned}
A_{mm} &= \frac{-4}{m\pi}\left(\frac{(-1)^m - 1}{m\pi} + \left[\frac{\cos(2m\pi x)}{4m\pi}\right]_0^1\right) \\
&= \frac{-4((-1)^m - 1)}{m^2\pi^2}
\end{aligned}$$

So the solution is

$$\begin{aligned}
u(x,y,t) = &\sum_{m \neq n} \frac{-4}{n\pi^2}\left(\frac{(-1)^m - 1}{m} + \frac{m((-1)^{n+m} - 1)}{n^2 - m^2}\right) e^{-\pi^2(m^2+n^2)t} \sin(m\pi x)\sin(n\pi y) \\
&+ \sum_{m=1}^{+\infty} \frac{-4((-1)^m - 1)}{m^2\pi^2} e^{-2\pi^2 m^2 t} \sin(m\pi x)\sin(m\pi y)
\end{aligned}$$