

### Sketch of solutions for the homework assignment 2.

Ex 2.3 #3 The function is even then  $b_n = 0$ . Then we must calculate:

$$\begin{aligned} a_0 &= \frac{1}{p} \int_0^p a(1 - (x/p)^2) dx = \frac{a}{p} \left( \int_0^p dx - (1/p^2) \int_0^p x^2 dx \right) \\ &= \frac{a}{p} (p - (1/p^2)(p^3/3)) \\ &= a(1 - 1/3) = 2a/3 \end{aligned}$$

Also

$$\begin{aligned} a_n &= \frac{2}{p} \int_0^p a(1 - (x/p)^2) \cos(n\pi x/p) dx \\ &= \frac{2a}{p} \left( \int_0^p \cos(n\pi x/p) dx - \int_0^p \left(\frac{x}{p}\right)^2 \cos(n\pi x/p) dx \right) \\ &= \frac{2a}{p} \left( \left[ \frac{\sin(n\pi x/p)}{(n\pi/p)} \right]_0^p - \int_0^1 X^2 \cos(n\pi X) p dX \right) \text{ substitution } X = x/p \\ &= -2a \int_0^1 X^2 \cos(n\pi X) p dX = \frac{-4a(-1)^n}{\pi^2 n^2} \end{aligned}$$

For the last line see Example 4 page 40. Then we have

$$a_n = \frac{4a(-1)^{n+1}}{(n\pi)^2}$$

Ex 2.3 #4 To give the Fourier series of  $f(x) = x^2$  on  $[-p, p]$  we use the previous exercise and the linearity of the Fourier series. First we put  $a = 1$  in the result of ex 3 to obtain:

$$\left( 1 - \left(\frac{x}{p}\right)^2 \right) = \frac{2}{3} + 4 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(n\pi)^2} \cos\left(\frac{n\pi}{p}x\right)$$

Then we remark that  $x^2 = \left( 1 - \left(\frac{x}{p}\right)^2 \right) \times (-p^2) + p^2$ . Then the Fourier series of  $x^2$  is given by

$$\begin{aligned} x^2 &= -p^2 \left( \frac{2}{3} + 4 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(n\pi)^2} \cos\left(\frac{n\pi}{p}x\right) \right) + p^2 \\ &= \frac{p^2}{3} - 4p^2 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(n\pi)^2} \cos\left(\frac{n\pi}{p}x\right) \end{aligned}$$

2.3 #20 (a) For any  $x$  we have

$$f_e(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_e(x)$$

then  $f_e$  is even. Also you can compute that  $f_o(-x) = -f_o(x)$  which means that  $f_o$  is odd.

(b) For any  $x$  we have

$$f_e(x) + f_o(x) = \frac{1}{2} [f(x) + f(-x) + f(x) - f(-x)] = f(x)$$

Uniqueness is proven at the back of the book.

(c) for any  $x$  we have

$$\begin{aligned} f_e(x+2p) &= \frac{f(x+2p) + f(-(x+2p))}{2} = \frac{f(x+2p) + f(-x-2p)}{2} \\ &= \frac{f(x) + f(-x)}{2} = f_e(x) \end{aligned}$$

because  $f(x+2p) = f(x)$  and  $f(-x-2p) = f(-x)$  as  $f$  is  $2p$ -periodic. Then  $f_e$  is  $2p$ -periodic. Same calculation for  $f_o$ .

(d) We have the Fourier series expansion of  $f$ :

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{+\infty} (a_n \cos(n\pi x/p) + b_n \sin(n\pi x/p)) \\ &= \left[ a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\pi x/p) \right] + \left[ \sum_{n=1}^{+\infty} b_n \sin(n\pi x/p) \right] \end{aligned}$$

You can check that the function in the left bracket is even and the one in the right bracket is odd. This means that this is a decomposition of  $f$  into an even and an odd function, but by the uniqueness of this decomposition proved in (b) the left bracket must be  $f_e$  and the right one is  $f_o$ , that is

$$\begin{cases} f_e(x) &= a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\pi x/p) \\ f_o(x) &= \sum_{n=1}^{+\infty} b_n \sin(n\pi x/p) \end{cases}$$

2.3 #23 The function is given by

$$f(x) = \begin{cases} -2x - 1 & \text{for } -1 \leq x \leq 0 \\ 1 & \text{for } 0 \leq x \leq 1 \end{cases}$$

Then we have for  $0 \leq x \leq 1$  ( $f_e$  is even so we will determine it only for positive values, the remainder will be obtained by symmetry):

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{1 + (-2(-x)) - 1}{2} = x$$

because  $0 \leq x \leq 1$  means that  $f(x) = 1$  and then  $-1 \leq -x \leq 0$  implying that  $f(-x) = -2(-x) - 1$ . Now by symmetry for  $-1 \leq x \leq 0$  we have

$$f_e(x) = f_e(-x) = -x$$

Then  $f_e(x) = |x|$ . Same for  $f_o$  gives you

$$f_o(x) = \begin{cases} -x - 1 & \text{for } -1 \leq x \leq 0 \\ 1 - x & \text{for } 0 \leq x \leq 1 \end{cases}$$

2.3 #25 Using the formula for Fourier coefficients of PW smooth function we have

Ex 2.5 #7 We use Parseval's Identity with  $f(x) = x/2$ . Then we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(x) dx &= a_0^2 + \frac{1}{2} \sum_{n=1}^{+\infty} (a_n^2 + b_n^2) \\ \iff \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{4} dx &= \frac{1}{2} \sum_{n=1}^{+\infty} \left( \frac{(-1)^{n+1}}{n} \right)^2 = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{1}{n^2} \end{aligned}$$

Because here  $a_0 = a_n = 0$  and  $b_n = \frac{(-1)^{n+1}}{n}$ . Also we have that

$$\int_{-\pi}^{\pi} \frac{x^2}{4} dx = \left[ \frac{x^3}{12} \right]_{-\pi}^{\pi} = \frac{\pi^3}{6}$$

So finally we find:

$$\frac{1}{2\pi} \frac{\pi^3}{6} = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$$