Sketch of solutions for the homework assignment 1.

Ex 2.1 #8 $f(x) = \cos(x) + \cos(\pi x)$.

(a) By remarking that $\cos(x) \leq 1$ and $\cos(\pi x) \leq 1$ for all x, we have that

$$f(x) = 2 \iff \cos x + \cos(\pi x) = 2 \iff \begin{cases} \cos x = 1 \\ \cos(\pi x) = 1 \end{cases}$$

The latter system has the following solutions: $x = 2k\pi$ for any integer k and $x = 2l$ for any integer l. The only value of k and l where those two coincide is when $l = k = 0$ that is $x = 0$, because of the irrationality of π (no integer multiple of π can be an integer).

This proves that the only solution to the equation $f(x) = 2$ is 0.

(b) Let's assume that f was periodic of period T. We will prove that this is impossible by finding a contradiction.

We know that $f(0) = 2$, then because $f(T) = f(0)$ (T-periodicity) then we also have $f(T) = 2$, meaning that we would have another solution to the equation $f(x) = 2$ which is impossible because of (a). Hence the contradiction, then f is not periodic.

Ex 2.1 #15 You have to prove two things: if F is 2π-periodic then $\int_0^{2\pi} f(t)dt = 0$ and the converse.

First let us assume that F is 2π-periodic and then prove that $\int_0^{2\pi} f(t)dt = 0$. Because F is 2π-periodic we have that $F(0) = F(2\pi)$ then this is

$$\int_a^0 f(t)dt = \int_a^{2\pi} f(t)dt \Rightarrow -\int_a^0 f(t)dt + \int_a^{2\pi} f(t)dt = 0$$
$$\Rightarrow \int_0^a f(t)dt + \int_a^{2\pi} f(t)dt = 0 \Rightarrow \int_0^{2\pi} f(t)dt = 0$$

To prove the converse: assume that $\int_0^{2\pi} f(t)dt = 0$. We want to prove that $F(x + 2\pi) = F(x)$ for any x. We calculate $F(x + 2\pi) - F(x)$:

$$F(x+2\pi) - F(x) = \int_a^{x+2\pi} f(t)dt - \int_a^x f(t)dt = \int_a^{x+2\pi} f(t)dt + \int_x^a f(t)dt$$
$$= \int_x^{x+2\pi} f(t)dt$$

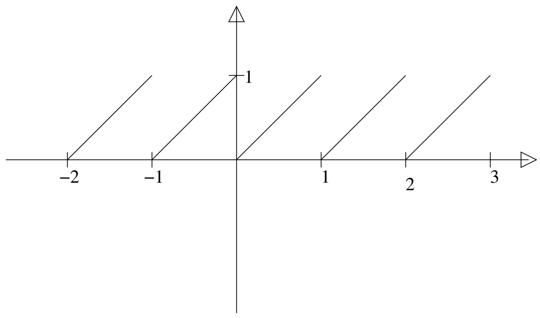
but as f is 2π-periodic we have (theorem 1) that

$$\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt = 0$$

then we have proved $F(x+2\pi) - F(x) = 0$ which means that

$$F(x+2\pi) = F(x)$$

2.1 #19 The case $p = 1$ (the other cases are just the same with a scale factor).
You have to break the study on intervals of length 1.
For example for $x \in [0,1)$ we have that $[x] = 0$ then $f(x) = x$.
For $x \in [1,2)$ we have that $[x] = 1$ then $f(x) = x - 1$.
More generally for $x \in [n, n+1)$ we have that $[x] = n$ then $f(x) = x - n$, where n is any integer.
Then the graph of f is
It is obvious on the graph that the period should be 1, still we must



prove it: take any x; remarking that $[x+1] = [x] + 1$ we have

$$f(x+1) = x + 1 - [x+1] = x + 1 - ([x] + 1) = x - [x] = f(x)$$

then f is 1-periodic (it is clear that 1 is the period).