

1. Sketch the graph and find the Fourier series of the following periodic functions (if the period is not given you have to determine it):

(a) The 2π -periodic function defined by: $f(x) = x^2$ for $-\pi \leq x \leq \pi$.

Classic: 2 integration by parts (or tabular integration) to compute the coefficients

(b) The function $g(x) = |\cos x|$.

The period of this function is π (so $p = \pi/2$) as we have

$$|\cos(\pi + x)| = |-\cos(x)| = |\cos x|$$

This function is even so $b_n = 0$. Then we compute (remembering that $\cos x \geq 0$ for $x \in [0, \pi/2]$):

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} |\cos x| dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x dx = \frac{2}{\pi} [\sin x]_0^{\pi/2} = \frac{2}{\pi}$$

and for a_n :

$$\begin{aligned} a_n &= \frac{4}{\pi} \int_0^{\pi/2} |\cos x| \cos(2nx) dx \\ &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \cos(2nx) dx \\ &= \frac{2}{\pi} \left[\frac{\sin(2n-1)x}{2n-1} + \frac{\sin(2n+1)x}{2n+1} \right]_0^{\pi/2} \\ &= \frac{2}{\pi} \left[\frac{\sin(2n-1)\pi/2}{2n-1} + \frac{\sin(2n+1)\pi/2}{2n+1} \right] \\ &= \frac{2}{\pi} \left[\frac{\sin(n\pi - \pi/2)}{2n-1} + \frac{\sin(n\pi + \pi/2)}{2n+1} \right] \\ &= \frac{2}{\pi} \frac{(-1)^{n+1}(2n+1) + (-1)^n(2n-1)}{4n^2 - 1} \\ &= \frac{4(-1)^{n+1}}{\pi(4n^2 - 1)} \end{aligned}$$

(c) $h(x) = 1/2 + \cos(2x)$

Already a Fourier series (period is π).

2. Find the sine and cosine series of the following function:

$f(x) = \sin 2x$ for $0 < x < \pi$

Already a Fourier sine series.

To compute the cosine series:

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin 2x dx = \frac{1}{\pi} \left[-\frac{\cos 2x}{2} \right]_0^{\pi} = 0$$

and for a_n , for $n \neq 2$:

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi \sin 2x \cos nx dx \\ &= \frac{1}{\pi} \left[\frac{\cos(n-2)x}{n-2} - \frac{\cos(n+2)x}{n+2} \right]_0^\pi \\ &= \frac{1}{\pi} \left[\frac{(-1)^{n-2} - 1}{n-2} - \frac{(-1)^{n+2} - 1}{n+2} \right] \\ &= \frac{4((-1)^n - 1)}{\pi(n^2 - 4)} \end{aligned}$$

For $n = 2$:

$$a_2 = \frac{2}{\pi} \int_0^\pi \sin 2x \cos 2x dx = \frac{1}{\pi} \int_0^\pi \sin 4x dx = 0$$

3. (a) Prove that for n a positive integer we have

$$\int_0^1 \sinh x \sin(n\pi x) dx = \frac{n\pi \sinh(1)}{1 + n^2\pi^2} (-1)^{n+1}$$

(Hint: you have to use 2 integration by parts)

We call $I_n = \int_0^1 \sinh x \sin(n\pi x) dx$ and we use two integration by parts:

$$\begin{aligned} I_n &= [\cosh x \sin(n\pi x)]_0^1 - \int_0^1 n\pi \cosh x \cos(n\pi x) dx \quad \left\{ \begin{array}{l} du = \sinh x dx \\ v = \sin(n\pi x) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} u = \cosh x \\ dv = n\pi \cos(n\pi x) dx \end{array} \right. \\ &= -n\pi \int_0^1 \cosh x \cos(n\pi x) dx \\ &= -n\pi \left([\sinh x \cos(n\pi x)]_0^1 + \int_0^1 n\pi \sinh x \sin(n\pi x) dx \right) \quad \left\{ \begin{array}{l} du = \cosh x dx \\ v = \cos(n\pi x) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} u = \sinh x \\ dv = -n\pi \sin(n\pi x) dx \end{array} \right. \\ &= -n\pi (\sinh(1)(-1)^n + n\pi I_n) \end{aligned}$$

Then we have

$$\begin{aligned} I_n &= -n^2\pi^2 I_n - n\pi \sinh(1)(-1)^n \\ \Leftrightarrow I_n(1 + n^2\pi^2) &= n\pi \sinh(1)(-1)^{n+1} \\ \Leftrightarrow I_n &= \frac{n\pi}{1 + n^2\pi^2} \sinh(1)(-1)^{n+1} \end{aligned}$$

(b) Give the solution to the wave equation (with $c = 1/\pi$ and $L = 1$) for the initial conditions $u(x, 0) = \sinh x$ and $\frac{\partial u}{\partial t}(x, 0) = -2 \sin(\pi x) + 4 \sin(5\pi x)$.

From (a) we have that

$$b_n = 2 \int_0^1 \sinh x \sin(n\pi x) dx = \frac{2n\pi}{1 + n^2\pi^2} \sinh(1)(-1)^{n+1}$$

We also have that $\lambda_n = cn\pi/L = n$. As $\frac{\partial u}{\partial t}(x, 0) = -2 \sin(\pi x) + 4 \sin(5\pi x)$ is already a Fourier series we can deduce that

$$b_1^* = \frac{1}{\lambda_1}(-2) = -2 \text{ and } b_5^* = \frac{1}{\lambda_5}4 = 4/5$$

Then the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2} \sinh(1)(-1)^{n+1} \cos nt \sin(n\pi x) - 2 \sin t \sin(\pi x) + \frac{4}{5} \sin 5t \sin(5\pi x)$$

4. Solve the wave equation (with $c = 1$ and $L = \pi$) with homogeneous boundary conditions for the following initial conditions:

$$u(x, 0) = -3 \sin x + \cos x \text{ and } \frac{\partial u}{\partial t}(x, 0) = 2 \sin 3x - \sin 5x.$$

First we have $\lambda_n = n$. We will first give the Fourier sine series of $\cos x$:

$$\begin{aligned} \text{For } n \neq 1, \beta_n &= \frac{2}{\pi} \int_0^\pi \cos x \sin(nx) dx \\ &= \frac{1}{\pi} \left[\frac{\cos((1-n)x)}{1-n} - \frac{\cos((n+1)x)}{n+1} \right]_0^\pi \\ &= \frac{1}{\pi} \left(\frac{\cos((1-n)\pi)}{1-n} - \frac{\cos((n+1)\pi)}{n+1} - \frac{1}{1-n} + \frac{1}{n+1} \right) \\ &= \frac{1}{\pi} \left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{1-n}}{1-n} + \frac{1}{n+1} - \frac{1}{1-n} \right) \\ &= \begin{cases} \frac{1}{\pi} \left(\frac{2}{n+1} + \frac{2}{n-1} \right) = \frac{4n}{\pi(n^2-1)} & \text{for } n \text{ even} \\ 0 & \text{for } n \neq 1 \text{ odd} \end{cases} \end{aligned}$$

$$\text{For } n = 1, \beta_1 = 2/\pi \int_0^\pi \sin x \cos x dx = 1/\pi \int_0^\pi \sin 2x dx = 0.$$

Using the fact that $-3 \sin x$ is already a Fourier sine series we have that

$$b_1 = -3 \text{ and } b_n = 0 \text{ for } n \neq 1 \text{ odd and } b_n = \frac{4n}{\pi(n^2-1)} \text{ for } n \text{ even}$$

Also as $2 \sin 3x - \sin 5x$ is already a Fourier sine series we have

$$b_3^* = 2/3 \text{ and } b_5^* = -1/5$$

Then the solution is

$$u(x, t) = -3 \cos t \sin x + \sum_{n \text{ even}} \frac{4n}{\pi(n^2-1)} \cos nt \sin nx + \frac{2}{3} \sin 3t \sin 3x - \frac{1}{5} \sin 5t \sin 5x$$