

1. Sketch the graph and find the Fourier series of the following periodic functions (if the period is not given you have to determine it):

(a) The  $2\pi$ -periodic function defined by:  $f(x) = 3x^2$  for  $-\pi \leq x \leq \pi$ .

Classic: 2 integration by parts (or tabular integration) to compute the coefficients

(b) The function  $g(x) = |\cos x|$ .

The period of this function is  $\pi$  (so  $p = \pi/2$ ) as we have

$$|\cos(\pi + x)| = |-\cos(x)| = |\cos x|$$

This function is even so  $b_n = 0$ . Then we compute (remembering that  $\cos x \geq 0$  for  $x \in [0, \pi/2]$ ):

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} |\cos x| dx = \frac{2}{\pi} \int_0^{\pi/2} \cos x dx = \frac{2}{\pi} [\sin x]_0^{\pi/2} = \frac{2}{\pi}$$

and for  $a_n$ :

$$\begin{aligned} a_n &= \frac{4}{\pi} \int_0^{\pi/2} |\cos x| \cos(2nx) dx \\ &= \frac{4}{\pi} \int_0^{\pi/2} \cos x \cos(2nx) dx \\ &= \frac{2}{\pi} \left[ \frac{\sin(2n-1)x}{2n-1} + \frac{\sin(2n+1)x}{2n+1} \right]_0^{\pi/2} \\ &= \frac{2}{\pi} \left[ \frac{\sin(2n-1)\pi/2}{2n-1} + \frac{\sin(2n+1)\pi/2}{2n+1} \right] \\ &= \frac{2}{\pi} \left[ \frac{\sin(n\pi - \pi/2)}{2n-1} + \frac{\sin(n\pi + \pi/2)}{2n+1} \right] \\ &= \frac{2}{\pi} \frac{(-1)^{n+1}(2n+1) + (-1)^n(2n-1)}{4n^2 - 1} \\ &= \frac{4(-1)^{n+1}}{\pi(4n^2 - 1)} \end{aligned}$$

(c)  $h(x) = 1/2 + \sin 3x - 3 \cos(4x)$

Already a Fourier series (period is  $2\pi$ ).

2. Find the sine and cosine series of the following function:

$f(x) = \cos 2x$  for  $0 < x < \pi$

Already a Fourier cosine series.

To compute the sine series: For  $n \neq 2$ :

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^\pi \cos 2x \sin nx dx \\
 &= \frac{1}{\pi} \left[ \frac{\cos(2-n)x}{2-n} - \frac{\cos(n+2)x}{n+2} \right]_0^\pi \\
 &= \frac{1}{\pi} \left[ \frac{(-1)^{2-n} - 1}{2-n} - \frac{(-1)^{n+2} - 1}{n+2} \right] \\
 &= \frac{2n((-1)^n - 1)}{\pi(n^2 - 4)}
 \end{aligned}$$

For  $n = 2$  we have:

$$b_2 = \frac{2}{\pi} \int_0^\pi \cos 2x \sin 2x dx = \frac{1}{\pi} \int_0^\pi \sin 4x dx = 0$$

3. (a) Prove that for  $n$  a positive integer we have

$$\int_0^1 \sinh x \sin(n\pi x) dx = \frac{n\pi \sinh(1)}{1 + n^2\pi^2} (-1)^{n+1}$$

(Hint: you have to use 2 integration by parts)

We call  $I_n = \int_0^1 \sinh x \sin(n\pi x) dx$  and we use two integration by parts:

$$\begin{aligned}
 I_n &= [\cosh x \sin(n\pi x)]_0^1 - \int_0^1 n\pi \cosh x \cos(n\pi x) dx \begin{cases} du = \sinh x dx \\ v = \sin(n\pi x) \end{cases} \Rightarrow \begin{cases} u = \cosh x \\ dv = n\pi \cos(n\pi x) dx \end{cases} \\
 &= -n\pi \int_0^1 \cosh x \cos(n\pi x) dx \\
 &= -n\pi \left( [\sinh x \cos(n\pi x)]_0^1 + \int_0^1 n\pi \sinh x \sin(n\pi x) dx \right) \begin{cases} du = \cosh x dx \\ v = \cos(n\pi x) \end{cases} \Rightarrow \begin{cases} u = \sinh x \\ dv = -n\pi \sin(n\pi x) \end{cases} \\
 &= -n\pi (\sinh(1)(-1)^n + n\pi I_n)
 \end{aligned}$$

Then we have

$$\begin{aligned}
 I_n &= -n^2\pi^2 I_n - n\pi \sinh(1)(-1)^n \\
 \Leftrightarrow I_n(1 + n^2\pi^2) &= n\pi \sinh(1)(-1)^{n+1} \\
 \Leftrightarrow I_n &= \frac{n\pi}{1 + n^2\pi^2} \sinh(1)(-1)^{n+1}
 \end{aligned}$$

(b) Give the solution to the wave equation (with  $c = 1/\pi$  and  $L = 1$ ) for the initial conditions  $u(x, 0) = \sinh x$  and  $\frac{\partial u}{\partial t}(x, 0) = -2 \sin(2\pi x) + 2 \sin(5\pi x)$ .

From (a) we have that

$$b_n = 2 \int_0^1 \sinh x \sin(n\pi x) dx = \frac{2n\pi}{1 + n^2\pi^2} \sinh(1)(-1)^{n+1}$$

We also have that  $\lambda_n = cn\pi/L = n$ . As  $\frac{\partial u}{\partial t}(x, 0) = -2\sin(2\pi x) + 2\sin(5\pi x)$  is already a Fourier series we can deduce that

$$b_2^* = \frac{1}{\lambda_2}(-2) = -1 \text{ and } b_5^* = \frac{1}{\lambda_5}2 = 2/5$$

Then the solution is

$$u(x, t) = \sum_{n=1}^{\infty} \frac{2n\pi}{1 + n^2\pi^2} \sinh(1)(-1)^{n+1} \cos nt \sin(n\pi x) - \sin 2t \sin(2\pi x) + \frac{2}{5} \sin 5t \sin(5\pi x)$$

4. Solve the wave equation (with  $c = 1$  and  $L = \pi$ ) with homogeneous boundary conditions for the following initial conditions:

$$u(x, 0) = -\sin x + \cos 2x \text{ and } \frac{\partial u}{\partial t}(x, 0) = 2\sin 5x - 3\sin 6x.$$

First we have  $\lambda_n = n$ . We will first give the Fourier sine series of  $\cos 2x$  (obtained in 2):

$$\beta_n = \begin{cases} \frac{2n((-1)^n - 1)}{\pi(n^2 - 4)} & \text{for } n \neq 2 \\ 0 & \text{for } n = 2 \end{cases}$$

For  $n = 1$ ,  $\beta_1 = -4/(-3\pi) = 4/(3\pi)$ .

Using the fact that  $-\sin x$  is already a Fourier sine series we have that

$$b_1 = \frac{4}{3\pi} - 1 \text{ and } b_n = 0 \text{ for } n \text{ even and } b_n = \frac{-4n}{\pi(n^2 - 4)} \text{ for } n \text{ odd } n \neq 1$$

Also as  $2\sin 5x - 3\sin 6x$  is already a Fourier sine series we have

$$b_5^* = 2/5 \text{ and } b_6^* = -3/6 = -1/2$$

Then the solution is

$$u(x, t) = (4/(3\pi) - 1) \cos t \sin x + \sum_{n \neq 1 \text{ odd}} \frac{-4n}{\pi(n^2 - 4)} \cos nt \sin nx + \frac{2}{5} \sin 5t \sin 5x - \frac{1}{2} \sin 6t \sin 6x$$