Mathematics 3150 Practice 2-Solution Summer 2004

1. Non-homogeneous heat problem.

   Find the solution of the heat equation $\dfrac{\partial u}{\partial t} = c^2 \Delta u$ in the unit square $(a = b = 1)$ with $c = 1/\pi$. The initial and boundary conditions are given by

$$f(x, y) = \sin \pi x \sin 2\pi y,\ f_1 = g_1 = g_2 = 0,\ f_2(x) = -\sin \pi x$$

Hint: you can use this formula

$$\int_0^1 \sin n\pi y \sinh \pi y dy = \frac{-n(-1)^n}{\pi(1 + n^2)} \sinh \pi$$

Solution:

First determine the steady state solution by solving the Dirichlet problem:

$$\begin{cases} \Delta v = 0 \\ \text{Boundary conditions given by } f_1 = g_1 = g_2 = 0,\ f_2(x) = -\sin \pi x \end{cases}$$

Then $A_n = C_n = D_n = 0$ and as f_2 is already a Fourier Sine series we have

$$B_1 = \frac{1}{\sinh \pi} \times (-1) = \frac{-1}{\sinh \pi} \text{ and } \forall n \neq 1, B_n = 0$$

Then the steady state is $v(x, y) = \dfrac{-1}{\sinh \pi} \sin \pi x \sinh \pi y$.

Then we solve the homogeneous Heat problem obtained by "substracting" the steady state to the original heat problem, that is we solve:

$$\begin{cases} \frac{\partial w}{\partial t} = c^2 \Delta w \\ \text{Initial condition: } f(x, y) - v(x, y) = \sin \pi x \sin 2\pi y + \frac{1}{\sinh \pi} \sin \pi x \sinh \pi y \\ \text{Zero boundary conditions: } f_1 = f_2 = g_1 = g_2 = 0 \end{cases}$$

Remember that A_{mn} is the coefficient of the double Fourier sine series of

$$\sin \pi x \sin 2\pi y + \frac{1}{\sinh \pi} \sin \pi x \sinh \pi y$$

and that this is linear, that is if we know α_{mn} the coefficients of $\sin \pi x \sin 2\pi y$ and β_{mn} those of $\sin \pi x \sinh \pi y$, then we have $A_{mn} = \alpha_{mn} + \dfrac{1}{\sinh \pi} \beta_{mn}$.

But as $\sin \pi x \sin 2\pi y$ is already a double Fourier sine series we have

$$\alpha_{1,2} = 1 \text{ and all other } \alpha_{mn}\text{'s are zero}$$

We calculate

$$\begin{aligned} \beta_{mn} &= 4 \int_0^1 \int_0^1 \sin \pi x \sinh \pi y \sin m\pi x \sin n\pi y dx dy \\ &= 4 \int_0^1 \sin \pi x \sin m\pi x dx \int_0^1 \sinh \pi y \sin n\pi y dy \end{aligned}$$

First we have

$$\int_0^1 \sin\pi x \sin m\pi x dx = \frac{1}{2}\int_0^1 (\cos(1-m)\pi x - \cos(1+m)\pi x)dx$$

Then for $m = 1$ we have

$$\int_0^1 \sin^2 \pi x dx = \frac{1}{2}\int_0^1 (1-\cos 2\pi x)dx = \frac{1}{2}$$

And for $m \neq 1$ we have

$$\int_0^1 \sin\pi x \sin m\pi x dx = \frac{1}{2}\left[\frac{\sin(1-m)\pi x}{(1-m)\pi} - \frac{\sin(1+m)\pi x}{(1+m)\pi}\right]_0^1 = 0$$

Then using the formula of the hint we find

$$\beta_{1n} = 4 \times \frac{1}{2} \times \frac{-n(-1)^n}{\pi(1+n^2)}\sinh\pi = \frac{-2n(-1)^n}{\pi(1+n^2)}\sinh\pi$$

and $\beta_{mn} = 0$ for $m \neq 1$. Then finally we obtain

$$\begin{cases} A_{1,2} = \alpha_{1,2} + \frac{1}{\sinh\pi}\beta_{1,2} = 1 + \frac{1}{\sinh\pi}\frac{-4}{\pi(1+4)}\sinh\pi = 1 - \frac{4}{5\pi} \\ A_{1n} = \alpha_{1n} + \frac{1}{\sinh\pi}\beta_{1n} = \frac{-2n(-1)^n}{\pi(1+n^2)} \\ \text{All other } A_{mn}\text{'s are zero} \end{cases}$$

Then the solution of the homogeneous Heat problem is

$$w(x,y,t) = \sin\pi x \sin 2\pi y e^{-5t} + \sum_{n=1}^{+\infty} \frac{-2n(-1)^n}{\pi(1+n^2)} \sin\pi x \sin n\pi y e^{(1+n^2)t}$$

because $\lambda_{1,2} = \sqrt{5}$ and $\lambda_{1n} = \sqrt{1+n^2}$. Finally the solution to the original problem is

$$u(x,y,t) = \sin\pi x \sin 2\pi y e^{-5t} + \sum_{n=1}^{+\infty} \frac{-2n(-1)^n}{\pi(1+n^2)} \sin\pi x \sin n\pi y e^{(1+n^2)t} - \frac{\sin\pi x \sinh\pi y}{\sinh\pi}$$

2. 2-dimensional wave equation.
   Solve the wave equation in the unit square ($a = b = 1$) and with $c = 1/\pi$. The initial conditions are given by

   $$f(x,y) = \sin\pi x \text{ and } g(x,y) = -2\sin 2\pi x \sin\pi y$$

   Solution:
   We calculate (using the calculation of the previous exercise)

   $$\begin{aligned} B_{mn} &= 4\int_0^1\int_0^1 \sin\pi x \sin m\pi x \sin n\pi y dx dy \\ &= 4\int_0^1 \sin\pi x \sin m\pi x dx \int_0^1 \sin n\pi y dy \\ &\Rightarrow \begin{cases} B_{1n} = 4 \times \frac{1}{2}\int_0^1 \sin n\pi y dy = 2\left[\frac{-\cos n\pi y}{n}\right]_0^1 = 2\frac{1-(-1)^n}{n} \\ B_{mn} = 0 \text{ for } m \neq 1 \end{cases} \end{aligned}$$

Then $B_{1n} = 4/n$ for n odd and $B_{1n} = 0$ for n even.
As $g(x, y)$ is already a Double Fourier sine series then we have

$$B^{\star}_{2,1} = \frac{1}{\lambda_{2,1}}(-2) = \frac{-2}{\sqrt{5}}$$

and all other $B^{\star}_{mn}$'s are zero. Then the solution is

$$u(x, y, t) = 4\sum_{k=0}^{+\infty} \frac{\cos\sqrt{1 + (2k+1)^2}t \sin \pi x \sin(2k+1)\pi y}{2k+1}$$
$$- \frac{2}{\sqrt{5}} \sin\sqrt{5}t \sin 2\pi x \sin \pi y$$

3. Wave equation in polar coordinates.
Solve the wave equation in the unit disk ($a = 1$), with $c = 1$. The initial conditions are given by (a)

$$f(r) = -J_0(\alpha_3 r) \text{ and } g(r) = \frac{J_0(\alpha_2 r)}{2}$$

Solution:
As $f(r)$ is already a Fourier-Bessel series we have

$$A_3 = -1 \text{ and all other } A_n\text{'s are zero}$$

Also $g(r)$ is already a Fourier-Bessel series we have

$$B_2 = \frac{1}{2\alpha_2} \text{ and all other } B_n\text{'s are zero}$$

Then the solution is

$$u(r, t) = -\cos\alpha_3 t J_0(\alpha_3 r) + \frac{1}{2\alpha_2} \sin\alpha_2 t J_0(\alpha_2 r)$$

(b)

$$f(r) = 1 - r^2 \text{ and } g(r) = 3J_0(\alpha_3 r)$$

Solution: As $g(r)$ is already a Fourier-Bessel Series we have

$$B_3 = \frac{3}{\alpha_3} \text{ and all others are zero}$$

For A_n we have

$$\begin{aligned} A_n &= \frac{2}{J_1^2(\alpha_n)} \int_0^1 (1 - r^2) J_0(\alpha_n r) r dr \\ &= \frac{2}{J_1^2(\alpha_n)} \frac{2}{\alpha_n^2} J_2(\alpha_n) \\ &= \frac{4J_2(\alpha_n)}{\alpha_n^2 J_1^2(\alpha_n)} \end{aligned}$$

Then the solution is

$$u(r, t) = \sum_{n=1}^{+\infty} \frac{4J_2(\alpha_n)}{\alpha_n^2 J_1^2(\alpha_n)} J_0(\alpha_n r) \cos\alpha_n t + \frac{3}{\alpha_3} J_0(\alpha_3 r) \sin\alpha_3 t$$