Mathematics 3150 Practice I-SOLUTIONS Summer 2005

1.(3 X 10 points) Sketch the graph and find the Fourier series of the following periodic functions (if the period is not given you have to determine it):

(a) The 2π-periodic function defined by: $f(x) = x$ for $-\pi \leq x \leq \pi$.

The function is odd so $a_0 = a_n = 0$. We calculate the b_n's using integration by parts:

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin(nx)dx = \frac{2}{\pi}\left[\frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos(nx)\right]_0^\pi$$
$$= \frac{2}{\pi}\frac{-\pi(-1)^n}{n} = 2\frac{(-1)^{n+1}}{n}$$

(b) The function $g(x) = |\sin x|$.

This function is π-periodic because:

$$|\sin(x+\pi)| = |-\sin x| = |\sin x|$$

Moreover it is even because $|\sin(-x)| = |-\sin x| = |\sin x|$. So $b_n = 0$. We calculate (this is not already a Fourier series!):

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2}|\sin x|dx = \frac{2}{\pi}\int_0^{\pi/2}\sin x dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi/2} = \frac{2}{\pi}$$

because for $x \in [0, \pi/2]$ $\sin x \geq 0$. Also we have

$$a_n = \frac{4}{\pi}\int_0^{\pi/2}\sin x\cos(2nx)dx = \frac{4}{\pi}\int_0^{\pi/2}\frac{1}{2}\left(\sin((2n+1)x) - \sin((1-2n)x)\right)dx$$
$$= \frac{2}{\pi}\left[-\frac{\cos((2n+1)x)}{2n+1} - \frac{\cos((2n-1)x)}{2n-1}\right]_0^{\pi/2}$$
$$= \frac{2}{\pi}\left(\frac{1}{2n+1} + \frac{1}{2n-1}\right)$$
$$= \frac{2}{\pi}\frac{2n-1+2n+1}{(2n)^2-1} = \frac{8n}{\pi(4n^2-1)}$$

(c) The 2π-periodic function defined by

$$h(x) = \begin{cases} x+\pi & \text{for } -\pi \leq x \leq -\pi/2 \\ \pi/2 & \text{for } -\pi/2 < x \leq \pi/2 \\ \pi - x & \text{for } \pi/2 < x \leq \pi \end{cases}$$

This function is even (easy check), so $b_n = 0$ and we calculate:

$$a_0 = \frac{1}{\pi}\left(\int_0^{\pi/2}\frac{\pi}{2}dx + \int_{\pi/2}^{\pi}(\pi - x)dx\right) = \frac{1}{\pi}\left(\frac{\pi^2}{4} + \frac{\pi^2}{8}\right) = \frac{3\pi}{8}$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\left(\int_0^{\pi/2} \frac{\pi}{2}\cos nx dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx dx\right) \\
&= \frac{2}{\pi}\left(\left[\frac{\pi}{2n}\sin nx\right]_0^{\pi/2} + \left[\frac{\pi}{n}\sin nx\right]_{\pi/2}^{\pi} - \left[\frac{1}{n^2}\cos(nx) + \frac{x}{n}\sin(nx)\right]_{\pi/2}^{\pi}\right) \\
&= \frac{2}{\pi}\left(\frac{\pi}{2n}\sin(n\pi/2) - \frac{\pi}{n}\sin(n\pi/2) - (\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\cos(n\pi/2) - \frac{\pi}{2n}\sin(n\pi/2)\right) \\
&= \frac{2}{\pi n^2}\left(-(-1)^n + \cos(n\pi/2)\right) \\
&= \begin{cases} \frac{2}{\pi(2k)^2}(-1+\cos k\pi) = \frac{2}{\pi(2k)^2}(-1+(-1)^k) & \text{for } n \text{ even, } n = 2k \\ \frac{2}{\pi n^2} & \text{for } n \text{ odd} \end{cases}
\end{aligned}
$$

So finally we have $a_{2k} = \frac{1}{2\pi k^2}(-1+(-1)^k)$ (even integers) and $a_{2k+1} = \frac{2}{\pi(2k+1)^2}$ (odd integers).

(d) $j(x) = \cos^2 x$

Use the trigonometric identity: $\cos^2 x = \frac{1}{2}(1+\cos(2x)$ and then this is already a Fourier series.

(e) The 2-periodic function defined by $k(x) = |x|$ for $-1 < x < 1$.

k is even so $b_n = 0$. Then we compute

$$
a_0 = \frac{1}{\pi}\int_0^{\pi} k(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx = \left[\frac{x^2}{2\pi}\right]_0^{\pi} = \frac{\pi}{2}
$$

2. Find the Fourier sin and cos series of the following functions:

(a) $f(x) = \sin x$ for $0 < x < \pi$.

f is already a sine series (as you can check by writing down the general sine series). To get the cosine series of f we will use the identity:

$$
\sin x\cos(nx) = \frac{1}{2}\left[\sin((n+1)x) - \sin((n-1)x)\right]
$$

We compute:

$$
a_n = \frac{2}{\pi}\int_0^{\pi} \sin x\cos(nx)dx = \frac{1}{\pi}\int_0^{\pi}\left[\sin((n+1)x) - \sin((n-1)x)\right]dx
$$

For $n = 1$ we have

$$
a_1 = \frac{1}{\pi}\left[-\frac{1}{2}\cos(2x)\right]_0^{\pi} = 0
$$

For $n \neq 1$ we have

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left[\left[-\frac{1}{n+1}\cos((n+1)x)\right]_0^{\pi} + \left[\frac{1}{n-1}\cos((n-1)x)\right]_0^{\pi}\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}(\cos((n+1)\pi) - 1) + \frac{1}{n-1}(\cos((n-1)\pi) - 1)\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}((-1)^{n+1} - 1) + \frac{1}{n-1}((-1)^{n-1} - 1)\right] \\
&= \begin{cases} \frac{1}{\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right] = \frac{-4}{\pi}\frac{1}{n^2-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

so the Fourier cosine series of $\sin x$ is

$$\sum_{k=1}^{+\infty} \frac{-4}{\pi} \frac{1}{(2k)^2-1} \cos(2kx)$$

(b) $g(x) = x^2$ for $0 < x < 1$.
For the sine series:

$$\begin{aligned}
b_n = 2\int_0^1 x^2 \sin(n\pi x)dx &= 2\left[\left[-\frac{x^2}{n\pi}\cos(n\pi x)\right]_0^1 + \frac{2}{n\pi}\int_0^1 x\cos(n\pi x)dx\right] \\
&= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2}{n\pi}\left[\left[\frac{x}{n\pi}\sin(n\pi x)\right]_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)dx\right]\right] \\
&= 2\left[\frac{(-1)^{n+1}}{n\pi} + \frac{2}{n^3\pi^3}((-1)^n - 1)\right]
\end{aligned}$$

For the cosine series:

$$a_0 = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3$$

$$\begin{aligned}
a_n = 2\int_0^1 x^2 \cos(n\pi x)dx &= 2\left[\left[\frac{x^2}{n\pi}\sin(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 x\sin(n\pi x)dx\right] \\
&= -\frac{4}{n\pi}\int_0^1 x\sin(n\pi x)dx \\
&= -\frac{4}{n\pi}\left[\left[-\frac{x}{n\pi}\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx\right] \\
&= \frac{4}{n^2\pi^2}(-1)^n
\end{aligned}$$

3. (a) Prove that for n a positive integer we have

$$\int_0^1 e^x \sin(n\pi x)dx = \frac{n\pi}{1+n^2\pi^2}\left(1 - (-1)^n e\right)$$

(Hint: you have to use 2 integration by parts)
We note

$$I_n = \int_0^1 e^x \sin(n\pi x)dx$$

We use a 2 integration by parts:

$$I_n = [e^x \sin(n\pi x)]_0^1 - \int_0^1 e^x n\pi \cos(n\pi x)dx \qquad \begin{cases} du = e^x dx \\ v = \sin(n\pi x) \end{cases} \Rightarrow \begin{cases} u = e^x \\ dv = n\pi\cos(n\pi x)dx \end{cases}$$

$$= -n\pi\int_0^1 e^x \cos(n\pi x)dx$$

$$= -n\pi\left([e^x\cos(n\pi x)]_0^1 + \int_0^1 e^x n\pi \sin(n\pi x)dx\right) \qquad \begin{cases} du = e^x dx \\ v = \cos(n\pi x) \end{cases} \Rightarrow \begin{cases} u = e^x \\ dv = -n\pi\sin(n\pi x)dx \end{cases}$$

$$= -n\pi\left(e(-1)^n - 1 + n\pi I_n\right)$$

Then we have

$$\begin{aligned} I_n &= -n^2\pi^2 I_n - n\pi\left(e(-1)^n - 1\right) \\ \Longleftrightarrow I_n(1+n^2\pi^2) &= n\pi\left(1 - e(-1)^n\right) \\ \Longleftrightarrow I_n &= \frac{n\pi}{1+n^2\pi^2}\left(1 - e(-1)^n\right) \end{aligned}$$

(b) Give the solution to the wave equation (with $c = 1/\pi$ and $L = 1$) for the initial conditions $u(x,0) = e^x$ and $\frac{\partial u}{\partial t}(x,0) = \sin\pi x - \sin 3\pi x$. We have (using (a)):

$$b_n = 2\int_0^1 e^x \sin(n\pi x)dx = \frac{2n\pi}{1+n^2\pi^2}\left(1 - e(-1)^n\right)$$

Now we calculate $b_n^\star$. We need the Fourier sine series of $\frac{\partial u}{\partial t}(x,0)$ but it is already a Fourier sine series so, because $\lambda_n = n$ we have

$$b_1^\star = 1,\ b_3^\star = -1/3 \text{ and } b_n^\star = 0 \text{ for all } n \neq 1, n \neq 3$$

Then the solution to the boundary value problem is

$$u(x,t) = \sum_{n=1}^{+\infty} \sin(n\pi x)\frac{2n\pi}{1+n^2\pi^2}\left(1 - e(-1)^n\right)\cos(nt) + \sin(\pi x)\sin(t) - \frac{1}{3}\sin(3\pi x)\sin(3t)$$

3.(2 X 25 Points) Solve the wave equation (with $c = 1$ and $L = \pi$) with homogeneous boundary conditions for the following initial conditions:

(a) $u(x,0) = \sin x - 4\sin 5x$ and $\frac{\partial u}{\partial t}(x,0) = 0$.
As $\frac{\partial u}{\partial t}(x,0) = 0$ all $b_n^\star$'s are zero.
$u(x,0)$ is already a Fourier sine series so we have $b_1 = 1$ and $b_5 = -4$ and other b_n's are zero. As $\lambda_n = n$ the solution is

$$u(x,t) = \sin x \cos t - 4\sin 5x \cos 5t$$

(b) $u(x,0) = -3\cos x$ and $\frac{\partial u}{\partial t}(x,0) = 2\sin 3x$.

$$\begin{aligned} b_n &= -3\frac{2}{\pi}\int_0^\pi \cos x \sin(nx)dx \\ &= -3\left(\frac{2}{\pi}\int_0^\pi \frac{1}{2}(\sin((n+1)x) - \sin((n-1)x)dx\right) \\ &= -\frac{3}{\pi}\left[-\frac{\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right]_0^\pi \\ &= -\frac{3}{\pi}\left(-\frac{\cos((n+1)\pi)}{n+1} + \frac{\cos((n-1)\pi)}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) \\ &= -\frac{3}{\pi}\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) \\ &= \begin{cases} \frac{-3}{\pi}\left(\frac{2}{n+1} - \frac{2}{n-1}\right) = \frac{12}{\pi(n^2-1)} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \end{aligned}$$

Now as $\frac{\partial u}{\partial t}(x,0)$ is already a sine series we have

$$b_3^\star = \frac{1}{3} \times 2 = \frac{2}{3}$$

and other $b_n^\star$'s are zero. So the solution is

$$u(x,t) = \sum_{k=1}^{+\infty} \frac{12}{\pi((2k)^2-1)} \sin(2kx)\cos(2kx) + \frac{2}{3}\sin 3x \sin 3t$$

(c) $u(x,0) = \sin x - \sin 3x$ and $\frac{\partial u}{\partial t}(x,0) = \sin x/2$.
The b_n are the Fourier sine series coefficients of $u(x,0)$; but $\sin x - \sin 3x$ is already a sine series then

$$b_1 = 1,\ b_3 = -1 \text{ and } b_n = 0 \text{ for all } n \neq 1, n \neq 3$$

We need to calculate the $b_n^\star$ ($\sin(x/2)$ is not already a Fourier sine series!):

$$\begin{aligned}
b_n^\star &= \frac{2}{n\pi}\int_0^\pi \sin(x/2)\sin(nx)dx \\
&= \frac{2}{n\pi}\int_0^\pi \frac{1}{2}(\cos((n-1/2)x) - \cos((n+1/2)x)dx \\
&= \frac{1}{n\pi}\left[\frac{\sin((n-1/2)x)}{n-1/2} - \frac{\sin((n+1/2)x}{n+1/2}\right]_0^\pi \\
&= \frac{1}{n\pi}2(\frac{\sin(n\pi-\pi/2)}{2n-1} - \frac{\sin(n\pi+\pi/2)}{2n+1} \\
&= \frac{-2}{n\pi}\left(\frac{(-1)^n}{2n-1} + \frac{(-1)^n}{2n+1}\right) \\
&= \frac{-2(-1)^n}{n\pi}\frac{4n}{(2n)^2-1} \\
&= \frac{8(-1)^{n+1}}{\pi(4n^2-1)}
\end{aligned}$$

Now as $\lambda_n = n$ the solution is

$$u(x,t) = \sin x \cos t - \sin(3x)\cos(3t) + \sum_{n=1}^{+\infty} \frac{8(-1)^{n+1}}{\pi(4n^2-1)}\sin(nx)\sin(nt)$$

(d) $u(x,0) = \cos x$ and $\frac{\partial u}{\partial t}(x,0) = \sin x$.
$\sin x$ is already a Fourier sine series then $b_1^\star = \frac{1}{\lambda_1} = 1$ because $\lambda_1 = 1$; and all other

$b_n^\star$'s are zero. We calculate the b_n's:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi \cos x \sin(nx) dx \\
&= \frac{2}{\pi}\int_0^\pi \frac{1}{2}(\sin((n+1)x) - \sin((n-1)x) dx \\
&= \frac{1}{\pi}\left[-\frac{\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right]_0^\pi \\
&= \frac{1}{\pi}\left(-\frac{\cos((n+1)\pi)}{n+1} + \frac{\cos((n-1)\pi)}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) \\
&= \frac{1}{\pi}\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) \\
&= \begin{cases} \frac{1}{\pi}\left(\frac{2}{n+1} - \frac{2}{n-1}\right) = \frac{-4}{\pi(n^2-1)} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}
\end{aligned}$$

Then the solution is:

$$u(x,t) = \sin x \sin t + \sum_{k=1}^{+\infty} \frac{-4}{\pi((2k)^2-1)} \sin(2kx)\cos(2kt)$$