Mathematics 2270 Homework 6-Sketch of solutions Fall 2004

4.1.5 Let us define $V = \{f \in \mathbb{R}_2[X] \mid \forall t \in \mathbb{R}, f(-t) = -f(t)\}$. The zero polynomial is in V. Moreover for $f, g \in V$ and $\lambda \in \mathbb{R}$ we have

$$\forall t \in \mathbb{R}, \begin{cases} (f+g)(-t) = f(-t) + g(-t) = -f(t) - g(t) = -(f+g)(t) \implies f+g \in V \\ (\lambda f)(-t) = \lambda f(-t) = -\lambda f(t) = -(\lambda f)(t) \implies \lambda f \in V \end{cases}$$

Then V is closed under addition and scaling, so V is a vector subspace.
Let us now find a basis of V. First we need to determine the polynomial in V. So let $f(t) = at^2 + bt + c \in \mathbb{R}_2[X]$ then

$$\begin{aligned} f \in V \iff \forall t \in \mathbb{R}, f(-t) = f(t) &\iff \forall t \in \mathbb{R}, a(-t)^2 - bt + c = -(at^2 + bt + c) \\ &\iff \forall t \in \mathbb{R}, at^2 - bt + c = -at^2 - bt - c \end{aligned}$$

So by identification we find that

$$f \in V \iff \begin{cases} a = -a \\ -b = -b \\ c = -c \end{cases} \iff a = c = 0 \iff f(t) = bt$$

Then $V = \text{span}\{t\}$ and then (t) is a basis of V (this is a non-zero vector so it is linearly independent).

4.1.8 The upper triangular matrices form a vector subspace T of the 3x3 matrices (usual checks are straightforward).
To find a basis, we just use the same idea as to do it for the full 3x3 matrices: we decompose any upper triangular matrix as follows

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then that means that T is spanned by those 6 matrices. To check that they are linearly independent we can use (for example) the definition. So we assume that $c_1, \ldots c_6 \in \mathbb{R}$ give a linear relation among those matrices:

$$c_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c_5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + c_6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

but the this is equivalent to

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ 0 & c_4 & c_5 \\ 0 & 0 & c_6 \end{pmatrix} = a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \iff c_1 = \cdots = c_6 = 0$$

then those matrices are linearly independent, then they form a basis of T.

4.1.58 a) $g \in V$ then

$$(g^2 + (g')^2)' = 2gg' + 2g'g'' = 2(gg' - gg') = 0 \text{ as } g'' = -g$$

so the function $g^2 + (g')^2$ is constant.
b) $g \in V$ and $g(0) = g'(0) = 0$. Then from a) we know that the function $g(x)^2 + (g'(x))^2 = c$ where c is a constant. But for $x = 0$ we have

$$c = g(0)^2 + (g'(0))^2 = 0 \text{ as we assumed } g(0) = g'(0) = 0$$

then for all x we have $g(x)^2 + (g'(x))^2 = 0$ but this implies that for all x, $g(x) = 0$.
c) $f \in V$. We define $g(x) = f(x) - f(0)\cos x - f'(0)\sin x$. We calulate

$$\begin{aligned} g'(x) = f'(x) + f(0)\sin x - f'(0)\cos x \implies g''(x) &= f''(x) + f(0)\cos x + f'(0)\sin x \\ &= -f(x) + f(0)\cos x \\ &= -g(x) \end{aligned}$$

then $g \in V$. Moreover you check that $g(0) = g'(0) = 0$ so applying b) we have that for all x, $g(x) = 0$. But this means that

$$\forall x \in \mathbb{R}, f(x) = f(0)\cos x + f'(0)\sin x$$

Then $V = \operatorname{span}\{\cos, \sin\}$.

4.2.53 We first compute $\ker T$. Let $f(t) = at^2 + bt + c \in \mathbb{R}_2[X]$, then

$$\begin{aligned} f \in \ker T \iff T(f) = 0 &\iff \forall t \in \mathbb{R}, f''(t) + 4f'(t) = 0 \\ &\iff \forall t \in \mathbb{R}, 2a + 4(2at + b) = 0 \\ &\iff \forall t \in \mathbb{R}, 8at + 4b + 2a = 0 \\ &\iff 8a = 0 \text{ and } 4b + 2a = 0 \iff a = b = 0 \iff f(t) = c \end{aligned}$$

Then $\ker T = \{c \mid c \in \mathbb{R}\}$, that is the kernel of T is the set of all constant polynomials. A basis of $\ker T$ is the $f(t) = 1$ (the constant polynomial equal to 1). Then $\dim \ker T = 1$, and using rank-nullity we have

$$\operatorname{rank} T = \dim \mathbb{R}_2[X] - \dim \ker T = 2$$

Now we can remark that $\operatorname{Im}(T) \subset \mathbb{R}_1[X]$ (the polynomials of degree 1 or less) as for all $f \in \mathbb{R}_2[X]$ we have

$$\deg(T(f)) = \deg(f'' + 4f') \leq \max\{\deg f'' + \deg f'\} = \max\{\deg f - 2, \deg f - 1\} \leq 1$$

as $\deg f \leq 2$. But we know that $\dim \mathbb{R}_1[X] = 2 = \dim \operatorname{Im}(T)$ then $\operatorname{Im}(T) = \mathbb{R}_1[X]$.

4.2.63 $\dim R^3 = 3$ and $\dim \mathbb{R}_3[X] = 4$ so there is no isomorphism between them (otherwise they would have the same dimension).

4.2.73 We first remark that $Z_n = \{a_n t^n + \cdots + a_1 t \mid a_n, \ldots, a_1 \in \mathbb{R}\}$. So a basis of Z_n is given by $(t, t^2, \ldots, t^n)$. Indeed they span Z_n and we already know that they are linearly independent. This also proves that $\dim Z_n = n = \dim \mathbb{R}_{n-1}[X]$. So T is an isomorphism iff $\ker T = \{0\}$. Let us compute it. Let $f(t) = a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{R}_{n-1}[X]$,

$$\begin{aligned} f \in \ker T \iff T(f) = 0 &\iff \forall t \in \mathbb{R}, \int_0^t f(x)dx = 0 \\ &\iff \forall t \in \mathbb{R}, \frac{a_{n-1}}{n}t^n + \frac{a_{n-2}}{n-1}t^{n-1} + \cdots + \frac{a_1}{2}t^2 + a_0 t = 0 \\ &\iff a_{n-1} = a_{n-2} = \cdots = a_1 = a_0 = 0 \iff f = 0 \end{aligned}$$

then $\ker T = \{0\}$, then T is an isomorphism.

4.2.74 You can check that the differential map gives an isomorphism between Z_n and $\mathbb{R}_{n-1}[X]$:

$$\begin{aligned} D : Z_n &\to \mathbb{R}_{n-1}[X] \\ f &\mapsto f' \end{aligned}$$

4.2.76 Using the linearity of T we have

$$T(0_V + 0_V) = T(0_V) + T(0_V)$$

but $0_V + 0_V = 0_V$ (neutral element of + in V) then we have $T(0_V + 0_V) = T(0_V)$ and then

$$T(0_V) + T(0_V) = T(0_V) \implies T(0_V) = 0_W$$

to obtain the last equality we have "simplified" by $T(0_V)$, that is we have added its opposite to both members of the equation.