Mathematics 2270 Homework 4-Sketch of solutions Fall 2004

3.3.10 Let's first find vectors in the kernel by finding linear relations among the columns:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}$$

It does not seem to have another one. We will calculate the dimensions of the image and the kernel. First we remark that

$$\operatorname{Im} A = \operatorname{span}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$$

But those 2 vectors are linearly independent (just row reduce the corresponding matrix, or use the definition), so they form a basis of $\operatorname{Im} A$, then $\operatorname{rank} A = \dim \operatorname{Im} A = 2$. But if we apply the rank-nullity theorem we have

$$\operatorname{rank} A + \dim \ker A = 3 \implies \dim \ker A = 1$$

But then as $\begin{pmatrix}1\\0\\0\end{pmatrix} \neq 0$ is in the kernel, it forms a basis (indeed a set of vectors consisting of only one non-zero vector is always linear independent).

3.3.14 Here we can just remark that $\operatorname{rank} A = 1$ because it is already reduced. Then using the rank-nullity theorem we have

$$\operatorname{rank} A + \dim \ker A = 3 \implies \dim \ker A = 2$$

Moreover as $\dim \operatorname{Im} A = \operatorname{rank} A = 1$ we need only one non-zero vector to form a basis, then we have (1) which forms a basis of $\operatorname{Im} A$.
Finding linear relations among the columns we find the following vectors in the kernel:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\-1\end{pmatrix}$$

Those two vectors are linearly independent because

$$\begin{pmatrix}1 & 0\\0 & 2\\0 & -1\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix} \implies \operatorname{rank}\begin{pmatrix}1 & 0\\0 & 2\\0 & -1\end{pmatrix} = 2$$

Then those two vectors form a basis of $\ker A$.

3.3.16 We row-reduce it to find its rank:

$$A = \begin{pmatrix}1 & 1 & 5 & 1\\0 & 1 & 2 & 2\\0 & 1 & 2 & 3\\0 & 1 & 2 & 4\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & 3 & -1\\0 & 1 & 2 & 2\\0 & 0 & 0 & 1\\0 & 0 & 0 & 2\end{pmatrix} \rightsquigarrow \begin{pmatrix}1 & 0 & 3 & 0\\0 & 1 & 2 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix}$$

then $\operatorname{rank} A = 3$. Then using the rank-nullity theorem we have

$$\dim \ker A + \operatorname{rank} A = 4 \implies \dim \ker A = 1$$

$$\operatorname{Im} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$$

As a side note, we know that $\operatorname{rank} A = \dim \operatorname{Im} A$ but $\operatorname{rank} A \leq 2$ so we deduce that we have at most two linear independent vectors among those spanning vectors.
We can build the following linear relations:

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0 \text{ and } -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 0$$

then we deduce that

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

so the last two vectors are redundant, we can discard them that is we have

$$\operatorname{Im} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

To check that we cannot go further we prove that those two vectors are linearly independent. We do it by finding the rank of the following matrix

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \det B = 2 - 1 = 1 \neq 0 \implies B \text{ is invertible} \implies \operatorname{rank} B = 2$$

then the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are linearly independent.

3.3.28 First we have 4 vectors in $\mathbb{R}^4$, so they will form a basis if and only if they are linearly independent as $\dim \mathbb{R}^4 = 4$. So we need to check for which values of k those vectors are independent. We build a matrix (4x4) which columns are those vectors, and find its rank:

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & k \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 4 & k-4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & k-13 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{pmatrix}$$

So if $k = 29$, the last line is full of zeros and then $\operatorname{rank} A < 4$ then the vectors are linearly dependendent. Moreover if $k \neq 29$ then the reduction can be carried on to its end

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and then $\operatorname{rank} A = 4$ and then the vectors are linearly independent. So those 4 vectors form a basis of $\mathbb{R}^4$ if and only if $k \neq 29$.

3.4.7 We have

$$x = 3v_1 + 4v_2$$

so $x \in \operatorname{span}\{v_1, v_2\}$ and $[x]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

3.4.9 There is no obvious relation so we use the row-reduced form to decide:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 1 & -1 \\ 4 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 0 & -1 \\ 0 & -4/3 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then $\operatorname{rank} A = 3$ and then x, v_1, v_2 are linearly independent therefore $x \notin \operatorname{span}\{v_1, v_2\}$.

3.4.21 First we have

$$S = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$$

First remark that $\det S = 1 + 6 = 7 \neq 0$, then S is invertible, which proves that $\mathcal{B}$. Now we have

$$S^{-1} = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$$

Then we can compute B:

$$B = S^{-1}AS = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 21 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}$$

Another way to see that is to calculate $T(v_1)$, $T(v_2)$ in the $\mathcal{B}$-coordinates:

$$\begin{cases} T(v_1) = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix} = 7v_1 \implies [T(v_1)]_{\mathcal{B}} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \\ T(v_2) = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies [T(v_2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

3.4.23 First we have

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

First remark that $\det S = 2 - 1 = 1 \neq 0$, then S is invertible, which proves that $\mathcal{B}$. Now we have

$$S^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Then we can compute B:

$$B = S^{-1}AS = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

Another way to see that is to calculate $T(v_1)$, $T(v_2)$ in the $\mathcal{B}$-coordinates:

$$\begin{cases} T(v_1) = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2v_1 \implies [T(v_1)]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ T(v_2) = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -v_2 \implies [T(v_2)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{cases}$$