

3.1.8 To find the kernel we solve the system $Ax = 0$. This system is

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \end{cases} \iff \begin{cases} x_1 = -x_2 - x_3 \\ x_2 = -2x_3 \end{cases} \iff \begin{cases} x_1 = x_3 \\ x_2 = -2x_3 \end{cases}$$

then we have

$$\ker A = \left\{ \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} \mid x_3 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

3.1.9 Here we can just remark that $\text{rank } A = 2$. Indeed we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then we know that $Ax = 0$ has unique solution (it is consistent because 0 is always a solution). This means that $\ker A = \{0\}$.

3.1.15 We know that we have

$$\text{Im } A = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$$

As a side note, we know that $\text{rank } A = \dim \text{Im } A$ but $\text{rank } A \leq 2$ so we deduce that we have at most two linear independent vectors among those spanning vectors.

We can build the following linear relations:

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 0 \quad \text{and} \quad -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 0$$

then we deduce that

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

so the last two vectors are redundant, we can discard them that is we have

$$\text{Im } A = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

To check that we cannot go further we prove that those two vectors are linearly independent. We do it by finding the rank of the following matrix

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \det B = 2 - 1 = 1 \neq 0 \implies B \text{ is invertible} \implies \text{rank } B = 2$$

then the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are linearly independent.

3.1.34 One geometric example is a projection on the plane P perpendicular to $L = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$.

Indeed if x is any vector in \mathbb{R}^3 , then we have the unique decomposition

$$x = x^{\parallel} + x^{\perp} \text{ where } x^{\parallel} \in P \text{ and } x^{\perp} \in L$$

And then $\text{proj}_P(x) = x^{\parallel}$. Then we have

$$x \in \ker \text{proj}_P \iff \text{proj}_P(x) = 0 \iff x^{\parallel} = 0 \iff x \in L$$

then $\ker \text{proj}_P = L$.

3.2.6 Let V and W two linear subspaces of \mathbb{R}^n .

a) We prove that $V \cap W$ is also a linear subspace of \mathbb{R}^n . First as $0 \in V$ and $0 \in W$ then $0 \in V \cap W$.

Let $x, y \in V \cap W$. Then in particular $x, y \in V$ then $x + y \in V$ as V is closed under addition. Similarly $x, y \in W$ then $x + y \in W$ as W is closed under addition; then $x + y \in V \cap W$. Also let $\lambda \in \mathbb{R}$. Then $\lambda x \in V$ as V is closed under scaling, and similarly $\lambda x \in W$; then $\lambda x \in V \cap W$. We have then proved that $V \cap W$ is a linear subspace of \mathbb{R}^n .

b) This is not true in general. We give here a counter-example: we take the two lines $L_1 = \text{span}\{e_1\}$ and $L_2 = \text{span}\{e_2\}$ in \mathbb{R}^2 . Those are 2 linear subspaces, but their union is not closed under addition: indeed $e_1 \in L_1$ and $e_2 \in L_2$ but $e_1 + e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in L_1 and neither in L_2 .

3.2.26 You can find the following linear relation among the columns:

$$2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then the vector $\begin{pmatrix} 2 \\ 3 \\ -1 \\ 0 \end{pmatrix}$ is in the kernel.

3.2.39 The answer is yes. To prove it we take a linear relation among the v_i 's and v :

$$c_1 v_1 + \cdots + c_m v_m + c v = 0 \tag{1}$$

and we prove that all the coefficients must be 0.

We first take care of c . We prove that it is zero by contradiction: let's assume that $c \neq 0$ then we can solve for v in (1) that is we have

$$v = -\frac{c_1}{c} v_1 - \cdots - \frac{c_m}{c} v_m$$

but this implies that $v \in \text{span}\{v_1, \dots, v_m\}$ which is false, hence the contradiction. Then $c = 0$.

But then now (1) becomes

$$c_1 v_1 + \cdots + c_m v_m = 0$$

which then implies that $c_1 = c_2 = \dots = c_m = 0$ because it is a linear relation among the v_i 's which are linearly independent. Thus we have proved that $c_1 = \dots = c_m = c = 0$, then the vectors v_1, \dots, v_m, v are linearly independent.