

2.3.34 a) The matrix $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ is invertible if and only if $rref(A) = I_3$. But if $a = 0$

or $b = 0$ or $c = 0$, then A contains a row consisting of zeroes, and then $rref(A) \neq I_3$, then A is non-invertible.

Moreover if $a \neq 0$ and $b \neq 0$ and $c \neq 0$, then multiplying by $1/a$, $1/b$ and $1/c$ the rows 1,2,3 respectively we find that $rref(A) = I_3$, then A is invertible.

So A is invertible if and only if $a \neq 0$ and $b \neq 0$ and $c \neq 0$. Moreover in that case we have

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix} = I_3$$

which means that $A^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$.

b) Let's do the same reasoning for a general n -dimensional diagonal matrix

$$A = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & a_n \end{pmatrix}$$

Here also A is invertible if and only if $rref(A) = I_n$. But if $a_i = 0$ for one $i \in \{1, \dots, n\}$, then the i th row of A contains only zeroes, and then $rref(A) \neq I_n$, then A is non-invertible.

Moreover if for all $i \in \{1, \dots, n\}$, $a_i \neq 0$ then multiplying by $1/a_i$ the i th row for all $i \in \{1, \dots, n\}$ we find that $rref(A) = I_n$, then A is invertible.

So we proved that A is invertible if and only if for all $i \in \{1, \dots, n\}$, $a_i \neq 0$.

Moreover in that case, by computing

$$A = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & a_n \end{pmatrix} \begin{pmatrix} 1/a_1 & 0 & 0 & \dots & 0 \\ 0 & 1/a_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1/a_n \end{pmatrix} = I_n$$

we deduce that

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 & 0 & \dots & 0 \\ 0 & 1/a_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1/a_n \end{pmatrix}$$

2.4.41 D_α and D_β represents the rotation R_α and R_β of angle α and β respectively, then $D_\alpha D_\beta$ represents $R_\alpha \circ R_\beta$; but for any vector \vec{x} , $R_\alpha \circ R_\beta(\vec{x}) = R_{\alpha+\beta}(\vec{x})$ (that is if you rotate the vector \vec{x} by R_β first and then rotate the resulting vector by R_α , it is the same as directly rotating \vec{x} by an angle of $\alpha + \beta$); so $R_\alpha \circ R_\beta = R_{\alpha+\beta}$. The same reasoning shows that $R_\beta \circ R_\alpha = R_{\alpha+\beta}$, that is $D_\beta D_\alpha$ represents also $R_{\alpha+\beta}$.
 b) We have

$$\begin{aligned} D_\alpha D_\beta &= \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{pmatrix} \end{aligned}$$

Which proves that $D_\alpha D_\beta$ represents the rotation $R_{\alpha+\beta}$ of angle $\alpha + \beta$; the same calculation holds for $D_\beta D_\alpha$.

2.4.45 Let $U : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $X : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be linear maps represented by S and B respectively.

We want to prove that $AS = B$. Let's look at the linear maps represented by AS and B . First the linear map represented by the matrix product AS is the composition $T \circ U$, by definition of the matrix product. So for $i \in \{1, \dots, m\}$ the i th column of AS is given by $T \circ U(\vec{e}_i)$. But as U is represented by S we know that $U(\vec{e}_i)$ is the i th column of S , that is \vec{v}_i . Then we have

$$T \circ U(\vec{e}_i) = T(U(\vec{e}_i)) = T(\vec{v}_i) = \vec{w}_i$$

So the i th column of AS is \vec{w}_i for all $i \in \{1, \dots, m\}$. But B has the same columns as AS , then $AS = B$. Now as S is invertible we can multiply both sides of $AS = B$ by S^{-1} (to the right) to obtain

$$A = BS^{-1}$$

This exercise proves that we can build a linear map from \mathbb{R}^m to \mathbb{R}^n just by giving the images of m arbitrary vectors $\vec{v}_1, \dots, \vec{v}_m$ (the only condition being that the matrix S having the \vec{v}_i 's as its columns must be invertible).

2.4.84 a) First the proof for $m = 2$, that is we will prove that all the entries of A^2 are less than or equal to r^2 .

We name a_{ij} the entry of A on the i th row and j th column, that is $A = (a_{ij})$.

To do this we want to apply Ex 83. First as the maximum column sum is r we then have

$$\forall j \in \{1, \dots, n\}, \sum_{i=1}^{i=n} a_{ij} \leq r$$

but as all the entries of A are positive or zero, then we have that for all i and j , $a_{ij} \leq r$. So we can now apply Ex. 83 to $A^2 = A.A$, with $s = r$ (and $B = A$): the conclusion is then that A^2 has all its entries being less than or equal to r^2 .

Let's now prove the general case by induction on the exponent m . We have already proved it for $m = 2$ (the first step, or initialization).

So now let's assume that for $m \geq 2$, we have that all entries of A^m are less than or equal to r^m . We must then prove that all entries of A^{m+1} are less than or equal to r^{m+1} . But as $A^{m+1} = A^m \cdot A$ we will once more apply exercise 83, now with " $A = A^m$ ", $s = r^m$ and $B = A$. The conclusion is then that A^{m+1} has all its entries less than or equal to $r^m \cdot r = r^{m+1}$. We have proved the "inductive" step.

Then this means that for all $m \geq 2$ we have proved that all entries of A^m are less than or equal to r^m .

b) You must remark here that $0 \leq r < 1$. Indeed r is the maximum column sum of A , but we know that all these columns sum must be less than 1, then so r must be. We denote by $a_{ij,m}$ the entry of A^m at the i th row and j th column (note that $a_{ij,m} \neq a_{ji}^m$, that is the entries of A^m are not the entries of A to the m th power).

So now using a) we have

$$\forall m \geq 2, \forall i, j \in \{1, \dots, n\}, 0 \leq a_{ij,m} \leq r^m \quad (1)$$

but as $0 \leq r < 1$, we know that $\lim_{m \rightarrow +\infty} r^m = 0$, then using (1) we find that $\lim_{m \rightarrow +\infty} a_{ij,m} = 0$ for all i, j , which means that $\lim_{m \rightarrow +\infty} A^m = 0$.

c) Let's denote the matrix sum $S = I_n + A + A^2 + \dots + A^m + \dots$, and call its entries S_{ij} . Then we have

$$S_{ij} = 1 + \sum_{s=1}^{+\infty} a_{ij,s}$$

This is a sum of non-negative real numbers, then using (1) once more and the fact that the power series $\sum_{s=0}^{+\infty} r^s$ converges (as $0 \leq r < 1$) then the series S_{ij} converges (comparison theorem for series with positive terms).

d) We compute and simplify

$$(I_n - A)(I_n + A + A^2 + \dots + A^m) = I_n + A + A^2 + \dots + A^m - A - A^2 - A^3 - \dots - A^{m+1} = I_n - A^{m+1}$$

Then if we let m tend to infinity in this equation we obtain (using the fact that $\lim_{m \rightarrow +\infty} A^{m+1} = 0$):

$$(I_n - A)(I_n + A + A^2 + \dots + A^m + \dots) = I_n$$

then this means that $I_n - A$ is invertible and

$$(I_n - A)^{-1} = I_n + A + A^2 + \dots + A^m + \dots$$