Mathematics 2270 Homework 2-Sketch of solutions Fall 2004

2.1.2 The matrix is

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}$$

2.1.3 The map is not linear because we have

$$2T\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) = 2\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} \text{ and } T\left(\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$$

that is $2T\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) \neq T\left(2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$.

2.1.5 The matrix is

$$\begin{pmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{pmatrix}$$

2.2.6 Let's denote the projection by proj_L. This is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$. Following the same calculations that for a projection in the plane we have:

$$\text{proj}_L(\vec{x}) = (\vec{x}.\vec{u})\vec{u}$$

for $\vec{u}$ a unit vector parallel to L. We need then to find a unit vector parallel to L, it is

$$\vec{u} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ because } \left\|\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right\| = 3$$

Then we have for $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$

$$\begin{aligned} \text{proj}_L(\vec{x}) = (\vec{x}.\vec{u})\vec{u} &= \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right)\frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \\ &= \frac{1}{9}(2x_1 + x_2 + 2x_3)\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 4x_1 + 2x_2 + 4x_3 \\ 2x_1 + x_2 + 2x_3 \\ 4x_1 + 2x_2 + 4x_3 \end{pmatrix} \end{aligned}$$

Then the matrix representing proj_L is

$$A = \frac{1}{9}\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

2.2.7 Let $\vec{x}, \vec{x}^{\perp}, \vec{x}^{||} \in \mathbb{R}^3$ such that $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp}$ with $\vec{x}^{\perp} \perp L$ and $\vec{x}^{||}$ is parallel to L. We know that then we have

$$ref_L(\vec{x}) = \vec{x}^{||} - \vec{x}^{\perp} = \vec{x}^{||} - (\vec{x} - \vec{x}^{||}) = 2\vec{x}^{||} - \vec{x} = 2\,\mathrm{proj}_L(\vec{x}) - \vec{x}$$

Indeed we know that $\mathrm{proj}_L(\vec{x}) = \vec{x}^{||}$. Then we use the previous exercises to compute

$$\mathrm{proj}_L\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \frac{1}{9}\begin{pmatrix}4 & 2 & 4\\2 & 1 & 2\\4 & 2 & 4\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{9}\begin{pmatrix}10\\5\\10\end{pmatrix}$$

Then we have

$$ref_L(\vec{x}) = \frac{1}{9}\begin{pmatrix}10\\5\\10\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1/9\\-4/9\\1/9\end{pmatrix}$$

2.2.15 We know (see previous exercises) that we have

$$ref_L(\vec{x}) = 2\,\mathrm{proj}_L(\vec{x}) - \vec{x}$$

Moreover we know that (because $\vec{u}$ is a unit vector parallel to L):

$$\mathrm{proj}_L(\vec{x}) = (\vec{x}.\vec{u})\vec{u} = (x_1u_1 + x_2u_2 + x_3u_3)\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}x_1u_1^2 + x_2u_1u_2 + x_3u_1u_3\\x_1u_1u_2 + x_2u_2^2 + x_3u_2u_3\\x_1u_1u_3 + x_2u_2u_3 + x_3u_3^2\end{pmatrix}$$

Then we have

$$ref_L(\vec{x}) = \begin{pmatrix}x_1u_1^2 + x_2u_1u_2 + x_3u_1u_3\\x_1u_1u_2 + x_2u_2^2 + x_3u_2u_3\\x_1u_1u_3 + x_2u_2u_3 + x_3u_3^2\end{pmatrix} - \begin{pmatrix}x_1\\x_2\\x_2\end{pmatrix} = \begin{pmatrix}x_1(u_1^2-1) + x_2u_1u_2 + x_3u_1u_3\\x_1u_1u_2 + x_2(u_2^2-1) + x_3u_2u_3\\x_1u_1u_3 + x_2u_2u_3 + x_3(u_3^2-1)\end{pmatrix}$$

then the matrix of ref_L is:

$$A = \begin{pmatrix}u_1^2-1 & u_1u_2 & u_1u_3\\u_1u_2 & u_2^2-1 & u_2u_3\\u_1u_3 & u_2u_3 & u_3^2-1\end{pmatrix}$$