

1.2 #20 The 2x2 matrices can only have rank 0,1,2 (as  $0 \leq \text{rank } A \leq 2$ ). Let's see how many different types you have for each rank.

First if A has rank 0 it must be the 0 matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So only one type here.

For rank one you have only the following two types (there must be only one leading one, and the row of zeroes can only be the last one to follow the rules of the reduced form):

$$T_1 = \begin{pmatrix} 1 & \star \\ 0 & 0 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

For rank 2 matrices there is only one type:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So overall there are only 4 types.

1.2 # 41 The moment is conserved so we have

$$\begin{aligned} m_1 \vec{v}_1 + m_2 \vec{v}_2 &= m_1 \vec{w}_1 + m_2 \vec{w}_2 \\ \Leftrightarrow \begin{cases} m_1 + 4m_2 &= 4m_1 + 2m_2 \\ m_1 + 7m_2 &= 7m_1 + 3m_2 \\ m_1 + 10m_2 &= 4m_1 + 8m_2 \end{cases} \\ \Leftrightarrow \begin{cases} -3m_1 + 2m_2 &= 0 \\ -6m_1 + 4m_2 &= 0 \\ -3m_1 + 2m_2 &= 0 \end{cases} \\ \Leftrightarrow -3m_1 + 2m_2 = 0 \text{ all 3 equations are equivalent} \\ \Leftrightarrow m_1 &= \frac{2}{3}m_2 \end{aligned}$$

1.3 #22 Let's call  $A \in M_3(\mathbb{R})$  the coefficient matrix of the system. The system have unique solution so the number of free variable must be 0 (else it would have either no solution or infinite number of solutions). But we have

$$0 = \# \text{ of free variables} = 3 - \text{rank } A$$

then the rank of A must be 3. Then we have  $\text{rref}(A) = I_3$  (because A is a 3x3 matrix of rank 3).

1.3 #23 By the same argument as in #22 the coefficient matrix must have rank 3. Then we must have

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Remark that then the right member must be of the form  $\begin{pmatrix} \star \\ \star \\ \star \\ 0 \end{pmatrix}$ .

- 1.3 #47 a.  $\vec{x} = \vec{0}$  is always a solution  
b. We have  $\text{rank } A \leq \text{number of equations} < \text{number of unknowns}$  so there are free variables. Moreover the system is consistent, then it has an infinite number of solutions.  
c. Let  $\vec{x}$  and  $\vec{y}$  be solutions of  $A\vec{x} = \vec{0}$ . Then we have

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}$$

that is  $\vec{x} + \vec{y}$  is a solution of the system.

- d. Similarly if  $\vec{x}$  is such that  $A\vec{x} = \vec{0}$  then  $A(k\vec{x}) = kA\vec{x} = \vec{0}$  so  $k\vec{x}$  is also a solution.

- 1.3 #48 a. Let  $\vec{x}_1$  and  $\vec{x}_h$  be vectors of  $\mathbb{R}^n$  such that  $A\vec{x}_1 = \vec{b}$  and  $A\vec{x}_h = \vec{0}$ . Then we have

$$A(\vec{x}_h + \vec{x}_1) = A\vec{x}_h + A\vec{x}_1 = \vec{b}$$

That is  $\vec{x}_1 + \vec{x}_h$  is a solution of  $A\vec{x} = \vec{b}$ .

- b.  $A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = \vec{0}$  then  $\vec{x}_1 - \vec{x}_2$  is a solution of  $A\vec{x} = \vec{0}$ .

- 1.3 #50 As  $\text{rank}(A | \vec{b}) = 4$  and it is a 4x4 matrix then we must have

$$\text{rref}(A | \vec{b}) = I_4 = \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

That means that the system is inconsistent (last row).