Exercise 4: First recall that $\mathbb{C}$ is $\mathbb{R}^2$. That is a complex number can be seen as a vector in $\mathbb{R}^2$ in the following way:

$$z \in \mathbb{C}, \text{ is viewed as the vector } \begin{pmatrix} \mathcal{R}(z) \\ \mathcal{I}(z) \end{pmatrix} \in \mathbb{R}^2 \quad \begin{pmatrix} \text{real and} \\ \text{imaginary} \\ \text{parts} \end{pmatrix}$$

So the vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ can be seen, conversely, as the complex number $a+ib \in \mathbb{C}$ (i is the root of -1 : $i^2 = -1$)

So $J : \mathbb{C} \to \mathbb{C}$ is then also a map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Moreover, the operations $+$ and scaling in $\mathbb{R}^2$ correspond to the addition of complex number and multiplication of a complex number by a real number:

in $\mathbb{C}$	in $\mathbb{R}^2$
$z = a+bi \in \mathbb{C}$, $z' = a'+ib' \in \mathbb{C}$	
$z+z'$: complex numbers addition	$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$
$\lambda \in \mathbb{R}$ $\quad \lambda z$: multiplication of z by the real number	$\lambda \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$

Now we can use the two ways of viewing $\mathbb{C}$ to prove that J is a linear map:

$$\forall z, z' \in \mathbb{C}, \forall \lambda \in \mathbb{R}, \left\{ \begin{array}{l} J(z+z') = \overline{z+z'} = \bar{z} + \bar{z}' = J(z) + J(z') \\ J(\lambda z) = \overline{\lambda z} = \bar{\lambda}\bar{z} = \lambda \bar{z} = \lambda J(z) \end{array} \right. \quad \left\{ \begin{array}{l} \lambda = \bar{\lambda} \\ \text{because } \lambda \in \mathbb{R} \end{array} \right.$$