

Mathematics 2270      PRACTICE EXAM I      Fall 2004

1. a) Find the rank of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

- b) Give the number of solutions for the following linear systems (you do not need to solve them! You can use part a)); explain:

$$A\vec{x} = \vec{b}$$

$$B\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$C\vec{z} = \vec{d}$$

2. Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the projection on the line parallel to  $\vec{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

- a) Find the matrix representing  $T$ .

- b) Show that  $\ker T = L$  where  $L$  is the line perpendicular to  $\vec{u}$ . Hint: use the formula  $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$

- c) Is  $T$  one-to-one?

3. Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  and  $U : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be two linear maps defined by

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ -x_2 \\ -x_1 \end{pmatrix} \qquad U \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1 + y_2 + y_3 \end{pmatrix}$$

- a) Write down the matrices  $A$  and  $B$  representing  $T$  and  $U$  respectively.

- b) Calculate the map  $U \circ T$  and the matrix  $C$  representing it.

- c) Show that  $U \circ T$  is a bijection.

- d) Show that for any  $\vec{z} \in \mathbb{R}^2$  it exists  $\vec{y} \in \mathbb{R}^3$  such that  $U(\vec{y}) = \vec{z}$ .

- e) Show that  $\ker T = \{0\}$ .

4. Let  $T : \mathbb{R}^m \rightarrow \mathbb{R}^p$  and  $U : \mathbb{R}^p \rightarrow \mathbb{R}^n$  be two linear maps and vectors  $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^p$  and  $\vec{w}_1, \dots, \vec{w}_m \in \mathbb{R}^n$  such that

$$\forall i \in \{1, \dots, m\}, T(\vec{e}_i) = \vec{v}_i \text{ and } U(\vec{v}_i) = \vec{w}_i$$

Give  $\text{Im}(U \circ T)$  as a span of vectors.

5. a) Show that  $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 - x_2 + 3x_3 = 0 \right\}$  is a linear subspace of  $\mathbb{R}^3$ .

- b) Show that  $W = \left\{ \begin{pmatrix} 2t + u \\ -t + 3u \end{pmatrix} \in \mathbb{R}^2 \mid t, u \in \mathbb{R} \right\}$  is a vector subspace of  $\mathbb{R}^2$ .

c) Show that  $X = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 - x_2 + 3x_3 = 1 \right\}$  is not a linear subspace of  $\mathbb{R}^3$

6. a) Tell if the vectors  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  are linearly independent or not.

b) Show that any vector  $\vec{x} \in \mathbb{R}^2$  can be expressed as a linear combination of the vectors  $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ . Moreover if  $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ , and if  $\vec{x} = \lambda\vec{u} + \mu\vec{v}$ , find  $\lambda, \mu$  in function of  $x_1, x_2$ .