Bivariate Splines and the Four Color Map Problem

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Annotated Slides:

http://www.math.utah.edu/~alfeld/talks/S13/4CMP.html

Outline of Talk

- **The Four Color Map problem**
- **Proof of the 4CMP**
- **Bivariate Spline Spaces**
- **Open Problems and Conjectures**
- **Generic Dimension**
- **Triangulations**
- **A simple proof for** S_4^1
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- **Subtriangulations**
- **Unavoidable Sets**
- **Discharging**
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- **Reducibility for** S_3^1
- **Another Unavoidable Set**
- **Remarks**

The Four Color Map Problem

Every map can be colored using only four colors such that no two neighboring countries are of the same color.

Long History:

- **1852** Francis Guthrie poses problem.
- **1878** Arthur Cayley presents problem to London Mathematical Society.
- **1879** Arthur Bray Kempe publishes "proof".
- **1890** Perey John Heawood points out flaw.
- **1976** Appel and Haken prove the Theorem.

What's a "map"?

- countries are connected
- neighbors share a line segment (not just a point)
- "normal maps": no country surrounds another, and no more than three countries meet in any one point.

Main reference: Appel and Haken: "Every Planar Map is Four Colorable", AMS, 1980.

Proof of the 4CMP

A "5-map" is a map requiring five colors.

Assume there is one.

- **If there's a 5-map then there is a smallest one.**
- **A configuration (submap) is "reducible" if a 5-map containing it can be reduced to a smaller 5-map.**
- **A set U of configurations is "unavoidable" if every 5-map contains at least one configuration in U.**
- **There exists an unavoidable set of reducible configurations $\Longrightarrow$ contradiction $\Longrightarrow$ there is no 5-map.**

Ideas

- **Use "discharging" to construct unavoidable sets.**
- **Dual graph: points for countries, edges for shared line segments — we have a triangulation!**
- **Ingenious ideas to prove reducibility.**
- **Use computers extensively.**
- **Many others.**

Bivariate Spline Spaces

$\mathcal{T}$ a triangulation.

$$S_d^r(\mathcal{T}) = \{s \in C^r(\Omega) : s|_\tau \in \mathbf{P}_d, \quad \forall \tau \in \mathcal{T}\}$$

where $\mathbf{P}_d$ is the $\binom{d+2}{2}$-dimensional space of bivariate polynomials of degree d.

Particular interest:

$$\boxed{d = 3, \qquad r = 1}$$

Open Problems

- **What is the dimension of S_3^1?**
- **Can one use S_3^1 to interpolate to function values at the vertices of T?**

Generic Dimension

Given r, d, and the number of interior and boundary vertices of a triangulation, S^r_d has a minimum dimension called the "generic" dimension.

If the dimension of $S^r_d(\mathcal{T})$ is not generic then an arbitrarily small perturbation of the location of the vertices of $\mathcal{T}$ will create a new triangulation $\tilde{\mathcal{T}}$ such that the dimension of $S^r_d(\mathcal{T})$ is the generic value.

An interior edge of $\mathcal{T}$ is "degenerate" (at its vertex v) if its two neighboring edges emanating from v are parallel.

A triangulation is "non-degenerate" if it contains no degenerate edges.

Known

The "generic" dimension of S^1_3 is $3V_B+2V_I+1$ (Billera and Whiteley).

Conjectures

(with the hope that we'll be able to prove some of the conjectures by constructing an "unavoidable set of reducible configurations".)

- **A triangulation is generic if it contains no degenerate edges**
- **Denoting by σ the number of "singular vertices" we have**

$$\boxed{\mathbf{dim} S_3^1 = 3V_B + 2V_I + 1 + \sigma}$$

- **Generically one can interpolate with S_3^1**
- **One can always interpolate with S_3^1.**

Remainder of Talk

Prove a dimension statement for S_1^4 instead of S_3^1 using the 4-color map techniques, and in the process comment on applications of those techniques to S_3^1.

Triangulations

Let

$$\mathcal{T} = \{\tau_i : i = 1, 2, \cdots, T\} \tag{1}$$

be a finite set of non-degenerate triangles in $\mathbb{R}^2$ and let

$$\Omega = \bigcup_{i=1}^{T} \tau_i. \tag{2}$$

Let

$$\mathcal{X} = \{x_i \in \Omega, i = 1, 2, \cdots, N\} \tag{3}$$

be the set of all vertices of triangles in $\mathcal{T}$. Then $\mathcal{T}$ is a *triangulation* **of Ω provided that**

1. **The interiors of the triangles in $\mathcal{T}$ are pairwise disjoint.**
2. **Each edge of a triangle in $\mathcal{T}$ is either on the boundary of Ω or else is a common edge of exactly two triangles in $\mathcal{T}$.**
3. **Each triangle in $\mathcal{T}$ contains no points of $\mathcal{X}$ other than its vertices.**
4. **Ω is homeomorphic to the unit square.**

Combinatorics of Triangulations

Let

$$\begin{aligned} V_B &= \textbf{number of boundary vertices in } \mathcal{T} \\ V_I &= \textbf{number of interior vertices in } \mathcal{T} \\ E_B &= \textbf{number of boundary edges in } \mathcal{T} \\ E_I &= \textbf{number of interior edges in } \mathcal{T} \\ N &= \textbf{number of triangles in } \mathcal{T} \end{aligned}$$

Then

$$\begin{aligned} E_B &= V_B \\ E_I &= V_B + 3V_I - 3 \\ N &= V_B + 2V_I - 2 \end{aligned}$$

Theorem. **Let $\mathcal{T}$ be a triangulations with no degenerate edges. Then**

$$\boxed{\textbf{dim} S_4^1(\mathcal{T}) = 6V - 3.}$$

Proof of dimS_4^1
(use s14.gif)

Subtriangulations

- **We will find reducible configurations.**
- **after removing a reducible configuration we may no longer have a triangulation.**
- **A "subtriangulation" of $\mathcal{T}$ is simply a subset of $\mathcal{T}$**

The dimension of S_4^1

We have to extend the dimension statement for S_4^1 to subtriangulations:

Theorem. **Let $\tilde{\mathcal{T}}$ be a subtriangulation of a triangulation $\mathcal{T}$ and suppose that $\mathcal{T}$ does not contain any degenerate edges. Then**

$$\boxed{\mathbf{dim} S_4^1(\tilde{\mathcal{T}}) = 3\tilde{V} + 3\tilde{N} + \tilde{E} - 2\tilde{E}_I.}$$

Note: In the special case that $\tilde{T}$ is actually a triangulation the formula reduces to the familiar statement

$$\mathbf{dim} S_4^1(\tilde{\mathcal{T}}) = 6V - 3.$$

Unavoidable Sets

- A "configuration" is a set of possible subtriangulations. We'll look at configurations sharing the same topology.
- Suppose there is a triangulation for which the dimension statement does not hold. Let's call it "offensive".
- If there is any offensive triangulation at all, there must be a smallest.
- A set of configurations is "unavoidable" if every offensive triangulation must contain at least one of the configurations.

Example

A "fill" and a "flap" form an unavoidable set. A "flap" is a triangle two of whose edge lie on the boundary of Ω, and a flap is a triangle with just one edge on the boundary.

Discharging

The "degree" of a vertex is the number of edges emanating from it.

Lemma. **Every triangulation contains at least one vertex of degree less than 6.**

Proof:

- **We assign to each vertex of Δ a "charge" of 6, and to each edge a charge of -2.**
- **The total charge is**

$$\boxed{\begin{aligned} 6(V_B + V_I) - 2E &= 6(V_B + V_I) - 2(2V_B + 3V_I - 3) \\ &= 2V_B + 6 \\ &> 0. \end{aligned}}$$

- **For each edge move half the charge to each endpoint.**
- **This process does not alter the total charge.**
- **For a vertex of degree k, the new charge is $6 - k$.**
- **The total charge is still positive.**
- **for some vertex $6 - k$ is positive, i.e., $k < 6$.**

An unavoidable set.

use slide unav.sv

Reducible Configurations

Let $\tilde{\mathcal{T}}$ and $\hat{T}$ be two subtriangulations where $\hat{\mathcal{T}}$ is obtained from $\tilde{\mathcal{T}}$ by removing a subtriangulation C. C is said to be "reducible" if

$$\begin{aligned}\mathbf{dim} S_4^1(\tilde{\mathcal{T}}) - \mathbf{dim} S_4^1(\hat{\mathcal{T}}) = \;& 3(\tilde{V} - \hat{V}) \\ & + 3(\tilde{N} - \hat{N}) \\ & + (\tilde{E} - \hat{E}) \\ & - 2(\tilde{E}_I - \hat{E}_I)\end{aligned}$$

Note: A fill is not reducible.

- **Clearly, an offensive subtriangulation cannot contain a reducible subtriangulation.**
- **Need to find an unavoidable set of reducible configurations.**
- **But: the argument about unavoidability applied to triangulations, not subtriangulations.**

Reducible Subtriangulations

To find a reducible subtriangulation of a subtriangulation $\tilde{T}$:

- **Add edges but not new vertices to $\tilde{\mathcal{T}}$ to obtain a triangulation Δ.**
- **Find a reducible configuration C of Δ.**
- **Consider the intersection of C and $\tilde{T}$**
- **It's not empty since we didn't introduce new vertices.**
- **We are done if:**
  - **We can find an unavoidable set U of reducible configurations for a triangulation.**
  - **We can show that every subtriangulation of a configuration in U is reducible.**

The unavoidable set constructed earlier satisfies these requirements!

Key ingredient: Every spline on $\mathcal{T}$ can be extended to a reducible subtriangulation.

Examples

On the following slides:

- **red indicates edges and vertices common to the reducible configuration and the subtriangulation, and Bézier ordinates implied directly by the spline outside the configuration.**
- **green indicates additional degrees of freedom, on the subtriangulation**
- **yellow indicates Bézier ordinates implied by smoothness conditions on the configuration.**
- $D\mathbf{dim} = \mathbf{dim} S^1_4(\tilde{\mathcal{T}}) - \mathbf{dim} S^1_4(\hat{\mathcal{T}})$
- $DN = \tilde{N} - \hat{N}$
- $DV = \tilde{V} - \hat{V}$
- $DE = \tilde{E} - \hat{E}$
- $DEI = \tilde{E}_I - \hat{E}_I$

In all cases

$$D\mathbf{dim} = 3DN + 3DV + DE - 2E_I$$

Note: the complete vertex stars require analysis of a linear system.

Example c1.gif

Example c2.gif

Example c3.gif

Example c4.gif

Reducible Configurations for S_3^1

- key point: Every spline can be extended from outside the configuration into the configuration.
- Suppose we remove a subtriangulation that is itself a triangulation.
- Expanding means force position and gradient at shared boundary vertices of the configuration and cross-boundary derivatives at midpoints of shared boundary edges. This leads to the necessary condition:

$$3V_B + 2V_I + 1 \geq 3V_B + E_I$$

Examples:

- A flap
- The star of an interior vertex of degree 3 (Clough-Tocher).
- The Morgan-Scott split

Of course, these three configurations do not form an unavoidable set.

An Unavoidable Set

Assume there are no flaps and no Clough-Tocher vertices. Consider the following discharge procedure:

- **Assign a charge of $60(6-k)$ to every vertex of degree k**
- **From every vertex of degree 4 or 5 move a charge of 30 to every neighboring vertex of degree > 5**
- **The total charge is positive (and equal to $120(V_B+3)$).**
- **This means there must be at least one of the following configurations in the triangulation. (A $\{4,5\}$ vertex is a vertex of degree 4 or 5).**

This gives rise to the following unavoidable set:

- **A 4-vertex with at least 1 $\{4,5\}$ neighbor**
- **A 5-vertex with at least 4 $\{4,5\}$ neighbors**
- **A 6-vertex with at least 1 $\{4,5\}$ neighbor**
- **A 7-vertex with at least 3 $\{4,5\}$ neighbors**
- **An 8-vertex with at least 5 $\{4,5\}$ neighbors**
- **A 9-vertex with at least 7 $\{4,5\}$ neighbors**
- **A 10-vertex with at least 9 $\{4,5\}$ neighbors**
- **An 11-vertex with 11 $\{4,5\}$ neighbors**

That set is to unwieldy. However, it can be reduced:

Covering Sets

- **An "$\{i,j\}$ edge formation" e_{ij} is a subtriangulation consisting of an edge connecting a vertex of degree i to a vertex of degree j together with all of the triangles that contain one of these two vertices.**
- **A formation is more general than a configuration since it may not have a unique topology.**
- **A 4-vertex with a $\{4,5\}$ neighbor implies we have e_{44} or e_{45}.**
- **An 8-vertex with 5 $\{4,5\}$ neighbors must have two $\{4,5\}$ neighbors that are themselves neighbors. So there must be one of e_{44}, e_{45}, or e_{55}.**
- **Using similar arguments we obtain the following unavoidable set:**

$$\{e_{44}, e_{45}, e_{46}, e_{47}, e_{55}, e_{56}, e_{57}\}$$

In these formations, one or both vertices may be on the boundary of the triangulation.

slide u1.gif

Remarks

- The appropriate dimension statement for S_3^1 on subtriangulations is

$$\boxed{\mathbf{dim} S_3^1 = 3V + N - E_I + \sigma.}$$

- A necessary condition for reducibility of a configuration is that

$$3V_B + 2V_I + 1 \geq 3V_S + E_S$$

where

$$E_S = \textbf{number of shared edges}$$

$$V_S = \textbf{number of shared vertices}$$

- if all edges and vertices are shared we have $E_S = V_S = V_B$ and so

$$\boxed{2V_I + 1 - V_B \geq 0.}$$

- We may have to construct reducible configurations with more than two interior vertices,
- or take into account boundary vertices.
- Unavoidability is the same as for the Four Color Map Problem.

- **Reducibility is very different. Checking reducibility may be hard to automate.**
- **If the conjectures are false this work may lead to a counter example.**
- **Any counter example is going to be complicated.**
- **For example, we know that a minimal counter example cannot contain flaps, Clough-Tocher vertices, or 4-4 or 4-5 edge formations.**