Multiscale Analysis of Heterogeneous Media in the Peridynamic Formulation

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Outline

1. Peridynamics and Continuum Applications.
Overview

The Classical Continuum Formulation.

The equation of motion (PDE)

\[ \rho(x) \partial_t^2 u(x, t) = \nabla \cdot \sigma(x, t) + b(x, t), \quad \sigma = g(\nabla u) \]

\( u \): Displacement field
\( \rho \): Mass density
\( \sigma \): Stress tensor
\( g \): Constitutive model
\( b \): External force
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The Peridynamic Formulation
(Silling 2000).

The equation of motion (PIDE)

\[ \rho(x) \partial_t^2 u(x, t) = \int_{H_\delta(x)} f(u'-u, x'-x) \, dx' + b(x, t), \quad u = u(x, t), \quad u' = u(x', t) \]
Overview

The Peridynamic Formulation
(Silling 2000).

The equation of motion (PIDE)

$$\rho(x) \frac{\partial^2 u(x,t)}{\partial t^2} = \int_{H_\delta(x)} f(u' - u, x' - x) \, dx' + b(x,t), \quad u = u(x,t), \quad u' = u(x',t)$$

- \(f\): Pairwise force (bond force). The force that \(x'\) exerts on \(x\).
- \(f\) depends on the relative position \(\xi = x' - x\), and the relative displacement \(\eta = u(x',t) - u(x,t)\).
- All constitutive information are included in \(f\).
- \(\delta\): Peridynamic horizon,

$$f(\eta, \xi) = 0, \quad \text{for } |\xi| > \delta.$$
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The equation of motion (PIDE)

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- The pairwise force satisfies

\[ f(-\eta, -\xi) = -f(\eta, \xi) \]

conservation of linear momentum

and

\[ f(\eta, \xi) \times (\eta + \xi) = 0 \]

conservation of angular momentum

which implies that

\[ f(\eta, \xi) = F(\eta, \xi)(\eta + \xi). \]
Example
The bond-stretch model

The bond force is given by

\[
f(\eta, \xi) = \begin{cases} 
    c \ s(\xi) \frac{\eta + \xi}{|\eta + \xi|}, & s(\xi) < s_0, \\
    0, & \text{otherwise.}
\end{cases}
\]

where

\[
s(\xi) = \frac{|\eta + \xi| - |\xi|}{|\xi|}
\]

(bond stretch)

Prototype brittle fracture
The Classical Formulation vs Peridynamics

**Classical Formulation**
- Particles interact directly only through contact forces.
- Partial differential equations. All fields \((u, \sigma, \ldots)\) are sufficiently smooth \((u \in W^{k,p})\).
- ‘Incompatible’ with fracture, e.g.,
  - Special techniques for modeling crack growth.
  - If the crack location is not known, the situation is difficult.

**Peridynamic Formulation**
- Nonlocal theory. Particles interact across a finite distance.
- Integral equations. Displacement field \(u\) can be discontinuous \((u \in L^p)\).
- Compatible with fracture (equation of motion holds everywhere in a body regardless of discontinuities).
- Under smoothness assumptions, as \(\delta \to 0\)
  \[\int_{H_\delta(x)} f(u(x') - u(x), x' - x) \, dx' \to \nabla \cdot \sigma(x)\]
  (Emmrich and Weckner 2007, linear case) and (Silling and Lehoucq 2008, general case).

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- Under smoothness assumptions, as $\delta \rightarrow 0$

\[
\int_{H_\delta(x)} f(u(x') - u(x), x' - x) \, dx' \rightarrow \nabla \cdot \sigma(x)
\]

(Emmrich and Weckner 2007, linear case) and (Silling and Lehoucq 2008, general case).
Continuum applications:
Dynamic fracture: Tearing of a membrane

Courtesy of Silling and collaborators, Sandia National Labs.
Continuum applications:
Interaction of 2 cracks: Peeling of a sheet

"Experimental data"

Courtesy of Silling and collaborators, Sandia National Labs.
Continuum applications: Cracking and fragmentation in glass

Courtesy of Silling and collaborators, Sandia National Labs.
Continuum applications:
Fragmentation of a concrete sphere

Courtesy of Silling and collaborators, Sandia National Labs.
Continuum applications:
Perforation of thin ductile targets

Courtesy of Silling and collaborators, Sandia National Labs.
Continuum applications:
Dynamic fracture in a balloon

Courtesy of Silling and collaborators, Sandia National Labs.
Modeling heterogeneous media using peridynamics

- Modeling fracture in homogeneous materials using peridynamics ✓.

Boeing is interested in fiber-reinforced materials. The length-scale of the fibers is very small relative to the structural length-scale. 'Impossible' to perform large scale simulations for the dynamics inside microstructured composites.

Homogenization and multiscale analysis.
Modeling heterogeneous media using peridynamics

- Modeling fracture in homogeneous materials using peridynamics ✔.

- What about composites (with microstructure)?
- Interested in modeling the dynamics inside heterogeneous materials using the peridynamic formulation.
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- Homogenization and multiscale analysis.
Multiscale analysis of fiber-reinforced materials using peridynamics

- The peridynamic equation for heterogeneous media (parameterized by $\varepsilon$).

$$\rho(x) \partial_t^2 u^\varepsilon(x, t) = \int_{H_x} f^\varepsilon (u^\varepsilon(x', t) - u^\varepsilon(x, t), x' - x, x) \, dx' + b^\varepsilon(x, t)$$

- Peridynamic models of fiber-reinforced materials.

1. A short-range bond force model.
2. A short-range + long-range bond force model.
3. A fluctuating long-range bond force model.
Fiber-reinforced materials

The peridynamic equation

- A unidirectional fiber-reinforced material.
- This material is periodic with period cell of side-length $\varepsilon$.

The peridynamic equation is given by

$$\rho(x) \partial_t^2 u^\varepsilon(x, t) = \int_{H_x} f^\varepsilon(u^\varepsilon(x', t) - u^\varepsilon(x, t), x' - x, x) \, dx' + b^\varepsilon(x, t), \quad (x, t) \in \Omega \times (0, T)$$

supplemented with initial data

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x),$$
$$\partial_t u^\varepsilon(x, 0) = v_0^\varepsilon(x).$$
Linear peridynamics

The bond force is given by

\[ f^\varepsilon(\eta^\varepsilon, \xi, x) = c^\varepsilon(x, x + \xi) \frac{\xi \otimes \xi}{|\xi|^3} \eta^\varepsilon, \quad \xi \in H_0 \]

where \( c^\varepsilon \) is a bounded real-valued function such that

- \( c^\varepsilon(x, x^\prime) = c^\varepsilon(x^\prime, x) \).
- \( c^\varepsilon(x, x^\prime) \) is \( \varepsilon \)-periodic in \( x \) and \( x^\prime \).
- The material properties are included in \( c^\varepsilon \).
- The three models are distinct in the way the coefficient \( c^\varepsilon \) and the neighborhood set \( H_0 \) are defined.

**Notation**

\[
\begin{align*}
\xi & = x^\prime - x, \\
\eta^\varepsilon & = u^\varepsilon(x^\prime, t) - u^\varepsilon(x, t).
\end{align*}
\]
Well-posedness of the peridynamic equation

\[
\begin{cases}
\partial_t^2 u^\varepsilon(x, t) = \int_{H_x} c^\varepsilon(x, x') \frac{(x' - x) \otimes (x' - x)}{|x' - x|^3} (u^\varepsilon(x', t) - u^\varepsilon(x, t)) \, dx' + b^\varepsilon(x, t) \\
u^\varepsilon(x, 0) = u_0^\varepsilon(x), \\
\partial_t u^\varepsilon(x, 0) = v_0^\varepsilon(x).
\end{cases}
\]

Existence and uniqueness

Let \( u_0^\varepsilon, v_0^\varepsilon \in L^p(\Omega)^3 \) and \( b^\varepsilon \in C([0, T]; L^p(\Omega)^3) \), where \( \frac{3}{2} < p \leq \infty \). Then \( u^\varepsilon \in C^2([0, T]; L^p(\Omega)^3) \) is the unique solution of the above peridynamic equation.
Multiscale analysis

\[
\begin{cases}
    \partial^2_t u^\varepsilon(x, t) = \int_{H_x} c^\varepsilon(x, x') \frac{(x' - x) \otimes (x' - x)}{|x' - x|^3} (u^\varepsilon(x', t) - u^\varepsilon(x, t)) dx' + b^\varepsilon(x, t) \\
    u^\varepsilon(x, 0) = u_0^\varepsilon(x), \\
    \partial_t u^\varepsilon(x, 0) = v_0^\varepsilon(x).
\end{cases}
\]

This \( \varepsilon \)-peridynamic equation is numerically very expensive.
Multiscale analysis

\[
\begin{aligned}
\partial_t^2 u^\varepsilon(x, t) &= \int_{H_x} c^\varepsilon(x, x') \frac{(x' - x) \otimes (x' - x)}{|x' - x|^3} (u^\varepsilon(x', t) - u^\varepsilon(x, t)) dx' + b^\varepsilon(x, t) \\
\begin{cases}
  u^\varepsilon(x, 0) &= u_0^\varepsilon(x), \\
  \partial_t u^\varepsilon(x, 0) &= v_0^\varepsilon(x).
\end{cases}
\end{aligned}
\]

This $\varepsilon$-peridynamic equation is numerically very expensive.

1. Homogenization
   The behavior of the averages of the solution $u^\varepsilon$ in the limit as $\varepsilon \rightarrow 0$

\[
\begin{align*}
  u^\varepsilon &\rightarrow u^H \quad \text{(weakly)}
\end{align*}
\]
Multiscale analysis

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\begin{aligned}
\partial_t^2 u^\varepsilon(x, t) &= \int_{H_x} c^\varepsilon(x, x') \frac{(x' - x) \otimes (x' - x)}{|x' - x|^3} (u^\varepsilon(x', t) - u^\varepsilon(x, t)) dx' + b^\varepsilon(x, t) \\
\quad \begin{aligned}
u^\varepsilon(x, 0) &= u_0^\varepsilon(x), \\
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\end{aligned}
\]

This \( \varepsilon \)-peridynamic equation is numerically very expensive.

1. Homogenization
   The behavior of the averages of the solution \( u^\varepsilon \) in the limit as \( \varepsilon \to 0 \)
   \[ u^\varepsilon \to u^H \text{ (weakly)} \]

2. Downscaling
   Find \( \hat{u}^\varepsilon \) such that
   \[ u^\varepsilon - \hat{u}^\varepsilon \to 0 \text{ (strongly)} \]

   and \( \hat{u}^\varepsilon \) is cheaper to compute than \( u^\varepsilon \).
A short-range bond force model

For $x$ and $x + \xi$ in $Y$, let

$$c(x, x + \xi) = \begin{cases} 
C_f, & \text{if } x \text{ and } x + \xi \text{ are in the fiber phase} \\
C_m, & \text{if } x \text{ and } x + \xi \text{ are in the matrix phase} \\
C_i, & \text{otherwise.}
\end{cases}$$

Extend $c$ periodically to $\mathbb{R}^3$. Then the bond force defined on $\Omega$ is given by

$$f_{\epsilon}(\eta_{\epsilon}, \xi, x) = \begin{cases} 
1/\epsilon^2 c(x/\epsilon, (x + \xi)/\epsilon) \xi \otimes \xi |\xi|^3 \eta_{\epsilon}, & |\xi| \leq \epsilon \delta_0 \\
\text{otherwise.}
\end{cases}$$
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\[
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Extend \(c\) periodically to \(\mathbb{R}^3\). Then the bond force defined on \(\Omega\) is given by

\[
f^\varepsilon(\eta^\varepsilon, \xi, x) = \begin{cases} 
  \frac{1}{\varepsilon^2} c \left( \frac{x}{\varepsilon}, \frac{x + \xi}{\varepsilon} \right) \frac{\xi \otimes \xi}{|\xi|^3} \eta^\varepsilon, & |\xi| \leq \varepsilon \delta \\
  0, & \text{otherwise}
\end{cases}
\]
A short-range bond force model

The peridynamic equation

\[
\begin{aligned}
\partial_t^2 u^\varepsilon(x, t) &= \int_{H_{\varepsilon\delta}(0)} c^\varepsilon(x, x+\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x, t)) d\xi + h(x, t) + R(x/\varepsilon), \\
u^\varepsilon(x, 0) &= u_0(x) + u_1(x/\varepsilon), \\
\partial_t u^\varepsilon(x, 0) &= v_0(x) + v_1(x/\varepsilon).
\end{aligned}
\]

Here \( c^\varepsilon(x, x+\xi) = \frac{1}{\varepsilon^2} c(x/\varepsilon, (x + \xi)/\varepsilon) \).
A short-range bond force model

The peridynamic equation

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\begin{aligned}
\partial_t^2 u^\varepsilon(x, t) &= \int_{H_{\varepsilon\delta}(0)} c^\varepsilon(x, x+\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x, t)) d\xi + h(x, t) + R(x/\varepsilon), \\
&\quad \text{subject to:} \\
&u^\varepsilon(x, 0) = u_0(x) + u_1(x/\varepsilon), \\
&\partial_t u^\varepsilon(x, 0) = v_0(x) + v_1(x/\varepsilon). 
\end{aligned}
\]

The homogenized equation

As \( \varepsilon \to 0 \),

\( u^\varepsilon \to u^H \) weakly in \( L^p(\Omega \times (0, T))^3 \),

where \( u^H \) solves

\[
\begin{aligned}
\partial_t^2 u^H(x, t) &= h(x, t), \\
u^H(x, 0) &= u_0(x), \\
\partial_t u^H(x, 0) &= v_0(x).
\end{aligned}
\]
A short-range bond force model

The peridynamic equation

\[
\begin{aligned}
\partial_t^2 u^\varepsilon(x, t) &= \int_{H^\varepsilon\delta(0)} c^\varepsilon(x, x+\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x, t)) d\xi + h(x, t) + R(x/\varepsilon), \\
u^\varepsilon(x, 0) &= u_0(x) + u_1(x/\varepsilon), \\
\partial_t u^\varepsilon(x, 0) &= v_0(x) + v_1(x/\varepsilon).
\end{aligned}
\]

Downscaling

Assume that \( h \in C([0, T]; C(\overline{\Omega})^3) \), and \( u_0 \) and \( v_0 \) are in \( C(\overline{\Omega})^3 \). Then for almost every \( t \in (0, T) \),

\[
\lim_{\varepsilon \to 0} \left\| u^\varepsilon(x, t) - (u^H(x, t) + r(x/\varepsilon, t)) \right\|_{L^p(\Omega)^3} = 0,
\]

where \( r \in C^2([0, T]; L^p_{per}(Y)^3) \) is the unique solution of

\[
\begin{aligned}
\partial_t^2 r(y, t) &= \int_{H_\delta(y)} c(y, y') \frac{(y' - y) \otimes (y' - y)}{|(y' - y)|^3} (r(y', t) - r(y, t)) \, dy' + R(y), \\
r(y, 0) &= u_1(y), \\
\partial_t r(y, 0) &= v_1(y).
\end{aligned}
\]
A short-range bond force model

The peridynamic equation

\[
\begin{cases}
\partial_t^2 u^\varepsilon(x, t) & = & \int_{H^\varepsilon\delta(0)} c^\varepsilon(x, x+\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x, t)) d\xi + h(x, t) + R(x/\varepsilon), \\
u^\varepsilon(x, 0) & = & u_0(x) + u_1(x/\varepsilon), \\
\partial_t u^\varepsilon(x, 0) & = & v_0(x) + v_1(x/\varepsilon).
\end{cases}
\]

Error estimate

Let \( t \in (0, T) \) and assume that \( u_0, v_0, \) and \( h(\cdot, t) \) are in \( C^{0,\beta}(\bar{\Omega})^3 \). Then

\[
\| u^\varepsilon(x, t) - (u^H(x, t) + r(x/\varepsilon, t)) \|_{L^p(\Omega)^3} \leq M(t)\varepsilon^\beta,
\]

where \( M(t) \) is independent of \( \varepsilon \).
A short-range bond force model

Can be shown, using two-scale convergence, that as \( \varepsilon \to 0 \),

\[
\int_{H_{\varepsilon \delta}(0)} c^{\varepsilon}(x, x+\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^{\varepsilon}(x+\xi, t) - u^{\varepsilon}(x, t)) d\xi \overset{\text{weak}}{\to}
\]

\[
\int_Y \int_{H_{\delta}(y)} c(y, y') \frac{(y' - y) \otimes (y' - y)}{|(y' - y)|^3} (r(y', t) - r(y, t)) \ dy' \ dy
\]

since \( f(\eta, y, y') = -f(-\eta, y', y) \)

\[
= - \int_Y \int_{H_{\delta}(y')} c(y', y) \frac{(y - y') \otimes (y - y')}{|(y - y')|^3} (r(y, t) - r(y', t))) \ dy \ dy'
\]

\[
= 0
\]
A short-range + long-range bond force model

\[ f^\varepsilon(\eta^\varepsilon, \xi, x) = \frac{\chi(\xi) c^\varepsilon(x, x + \xi) \xi \otimes \xi}{|\xi|^3} \eta^\varepsilon \]
A short-range + long-range bond force model

\[ f^\varepsilon(\eta^\varepsilon, \xi, x) = \left( \chi(\xi) \frac{c^\varepsilon(x, x + \xi)}{H_{\varepsilon\delta}(0)} + \chi(\xi) \frac{\lambda(\xi)}{H_{\gamma}(0)} \right) \frac{\xi \otimes \xi}{|\xi|^3} \eta^\varepsilon \]
A short-range + long-range bond force model

\[ f^\varepsilon(\eta^\varepsilon, \xi, x) = \left( \chi(\xi) c^\varepsilon(x, x + \xi) + \chi(\xi) \lambda(\xi) \right) \frac{\xi \otimes \xi}{|\xi|^3} \eta^\varepsilon \]

where,

\[ \lambda(\xi) = \begin{cases} 
  C_M^f |\xi|, & \text{if } \xi \text{ is parallel to the fiber direction} \\
  C_M^m |\xi|, & \text{otherwise} 
\end{cases} \]

and \( \gamma, C_M^f, C_M^m \) are macroscopic parameters.
A short-range + long-range bond force model

The peridynamic equation

\[
\begin{aligned}
\partial_t^2 u^\varepsilon(x, t) &= \int_{H_{\varepsilon\delta}(0)} c^\varepsilon(x, x+\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x, t)) d\xi \\
&\quad + \int_{H_{\gamma}(0)} \lambda(\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi, t) - u^\varepsilon(x, t)) d\xi + h(x, t) + R(x/\varepsilon), \\
\end{aligned}
\]

\[
\begin{aligned}
u^\varepsilon(x, 0) &= u_0(x) + u_1(x/\varepsilon), \\
\partial_t u^\varepsilon(x, 0) &= v_0(x) + v_1(x/\varepsilon).
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A short-range + long-range bond force model

The peridynamic equation

\[
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\partial_t^2 u^\varepsilon(x,t) &= \int_{H_\varepsilon(0)} c^\varepsilon(x,x+\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi,t) - u^\varepsilon(x,t)) d\xi \\
&+ \int_{H_\gamma(0)} \lambda(\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^\varepsilon(x+\xi,t) - u^\varepsilon(x,t)) d\xi + h(x,t) + R(x/\varepsilon),
\end{aligned}
\]

\[u^\varepsilon(x,0) = u_0(x) + u_1(x/\varepsilon), \quad \partial_t u^\varepsilon(x,0) = v_0(x) + v_1(x/\varepsilon).\]

The homogenized equation

As \(\varepsilon \to 0\),

\[u^\varepsilon \to u^H \text{ weakly in } L^p(\Omega \times (0,T))^3,\]

where \(u^H\) solves

\[
\begin{aligned}
\partial_t^2 u^H(x,t) &= \int_{H_\gamma(0)} \lambda(\xi) \frac{\xi \otimes \xi}{|\xi|^3} (u^H(x+\xi,t) - u^H(x,t)) d\xi + h(x,t), \\
u^H(x,0) &= u_0(x), \quad \partial_t u^H(x,0) = v_0(x).
\end{aligned}
\]
A short-range + long-range bond force model

The peridynamic equation

\[
\begin{cases}
\partial_t^2 u^\varepsilon(x, t) = \int_{H_{\varepsilon\delta}(0)} c^\varepsilon(x, x+\zeta) \frac{\xi \otimes \xi}{|\xi|^3} \left( u^\varepsilon(x+\zeta, t) - u^\varepsilon(x, t) \right) d\zeta \\
\quad + \int_{H_{\gamma}(0)} \lambda(\zeta) \frac{\xi \otimes \xi}{|\xi|^3} \left( u^\varepsilon(x+\zeta, t) - u^\varepsilon(x, t) \right) d\zeta + h(x, t) + R(x/\varepsilon), \\
\quad u^\varepsilon(x, 0) = u_0(x) + u_1(x/\varepsilon), \quad \partial_t u^\varepsilon(x, 0) = v_0(x) + v_1(x/\varepsilon).
\end{cases}
\]

Downscaling

\[
\lim_{\varepsilon \to 0} \left\| u^\varepsilon(x, t) - \left( u^H(x, t) + r(x/\varepsilon, t) \right) \right\|_{L_p(\Omega)^3} = 0,
\]

where \( r \in C^2([0, T]; L_p^{\text{per}}(Y)^3) \) is the unique solution of

\[
\begin{cases}
\partial_t^2 r(y, t) = \int_{H_{\delta}(y)} c(y, y') \frac{(y' - y) \otimes (y' - y)}{|(y' - y)|^3} \left( r(y', t) - r(y, t) \right) dy' \\
\quad - \int_{H_{\gamma}(0)} \lambda(\xi) \frac{\xi \otimes \xi}{|\xi|^3} d\xi \ r(y, t) + R(y), \\
\quad r(y, 0) = u_1(y), \quad \partial_t r(y, 0) = v_1(y).
\end{cases}
\]
A fluctuating long-range bond force model

The bond force is given by

\[ f^\epsilon(\eta^\epsilon, \xi, x) = \begin{cases} 
  c^\epsilon(x, x + \xi) \frac{\xi \otimes \xi}{|\xi|^3} \eta^\epsilon, & |\xi| \leq \delta \\
  0, & \text{otherwise.} 
\end{cases} \]
A fluctuating long-range bond force model

The bond force is given by

\[ f_{\varepsilon}^{\varepsilon}(\eta^{\varepsilon}, \xi, x) = \begin{cases} 
    c_{\varepsilon}^{\varepsilon}(x, x + \xi) \frac{\xi \otimes \xi}{|\xi|^3} \eta^{\varepsilon}, & |\xi| \leq \delta \\
    0, & \text{otherwise.}
\end{cases} \]

where \( c_{\varepsilon} \) is \( \varepsilon Y \)-periodic and given by

\[ c_{\varepsilon}^{\varepsilon}(x, x + \xi) = \begin{cases} 
    C_{f}|\xi|, & \text{if } x \text{ and } x + \xi \text{ are in the fiber phase,} \\
    \varepsilon C_{m}|\xi|, & \text{and } \xi \text{ is parallel to the fiber direction,} \\
    0, & \text{otherwise.}
\end{cases} \]
A fluctuating long-range bond force model

The peridynamic initial value problem
(second-order ACP in $L^p(\Omega)^3$, $1 \leq p < \infty$)

\[
\begin{aligned}
\ddot{u}\varepsilon(t) &= A\varepsilon u\varepsilon(t), \quad t \in [0, T] \\
u\varepsilon(0) &= u^0, \\
\dot{u}\varepsilon(0) &= v^0.
\end{aligned}
\]

For $v \in L^p(\Omega)^3$,

\[
A\varepsilon = \chi_f\varepsilon A_f + \varepsilon(A_m - C_m/C_f\chi_f\varepsilon A_f)
\]

\[
A_m v(x) = C_m \int_{H_\delta(0)} \frac{\xi \otimes \xi}{|\xi|^2} (v(x + \xi) - v(x))d\xi
\]

\[
A_f v(x) = C_f \int_{I_\delta(0)} \frac{\xi \otimes \xi}{|\xi|^2} (v(x + \xi) - v(x))d\xi
\]

Here $\chi_f(x) = \chi_f(x/\varepsilon)$. 

The homogenized equation

For $t \in (0, T)$ and as $\varepsilon \to 0$,

\[
u\varepsilon(t) \to u^H(t)
\]

weakly in $L^p(\Omega)^3$,

where $u^H$ solves

\[
\begin{aligned}
\ddot{u}^H(t) &= A_f u^H(t) + h(t), \quad t \in [0, T] \\
u^H(0) &= u^0, \\
\dot{u}^H(0) &= v^0.
\end{aligned}
\]

and $h(t) = (\theta_f - 1)A_f(u_0 + tv_0)$. 

Trotter
A fluctuating long-range bond force model

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\]

The homogenized equation

For $t \in (0, T)$ and as $\varepsilon \to 0$,

\[u^\varepsilon(t) \to u^H(t) \text{ weakly in } L^p(\Omega)^3,
\]

where $u^H$ solves

\[
\begin{cases}
\ddot{u}^H(t) = A_f u^H(t) + h(t), & t \in [0, T] \\
u^H(0) = u^0, \\
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\end{cases}
\]

and

\[h(t) = (\theta_f - 1)A_f(u^0 + tv^0).
\]
A fluctuating long-range bond force model

The peridynamic initial value problem
(second-order ACP in $L^p(\Omega)^3$, $1 \leq p < \infty$)

\[
\begin{align*}
\ddot{u}^\varepsilon(t) &= A^\varepsilon u^\varepsilon(t), \quad t \in [0, T] \\
 u^\varepsilon(0) &= u^0, \\
 \dot{u}^\varepsilon(0) &= \nu^0.
\end{align*}
\]

The homogenized equation

For $t \in (0, T)$ and as $\varepsilon \rightarrow 0$,

\[u^\varepsilon(t) \rightarrow u^H(t) \text{ weakly in } L^p(\Omega)^3,\]

where $u^H$ solves

\[
\begin{align*}
\ddot{u}^H(t) &= A_f u^H(t) + h(t), \quad t \in [0, T] \\
 u^H(0) &= u^0, \\
 \dot{u}^H(0) &= \nu^0.
\end{align*}
\]

and

\[h(t) = (\theta_f - 1) A_f (u^0 + t\nu^0).\]
Trotter-Kato Theorem and weak convergence of generators of semigroups

\[ u^\varepsilon(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} (A^\varepsilon)^n u^0 + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (A^\varepsilon)^n v^0 \]
Trotter-Kato Theorem and weak convergence of generators of semigroups

For $n = 1,2,3,\ldots$, as $\varepsilon \to 0$,

$$(A^\varepsilon)^n v \to \theta_f(A_f)^n v \text{ weakly in } L^p(\Omega)^3.$$ 

$$u^\varepsilon(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} (A^\varepsilon)^n u^0 + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (A^\varepsilon)^n v^0$$
Trotter-Kato Theorem and weak convergence of generators of semigroups

For \( n = 1, 2, 3, \ldots \), as \( \varepsilon \to 0 \),

\[(A^\varepsilon)^n v \to \theta_f(A_f)^n v \text{ weakly in } L^p(\Omega)^3.\]

\[
u^\varepsilon(t) = u^0 + tv^0 + \sum_{n=1}^\infty \frac{t^{2n}}{(2n)!} (A^\varepsilon)^n u^0 + \sum_{n=1}^\infty \frac{t^{2n+1}}{(2n+1)!} (A^\varepsilon)^n v^0\]
Trotter-Kato Theorem and weak convergence of generators of semigroups

For \( n = 1, 2, 3, \ldots \), as \( \varepsilon \to 0 \),

\[
(A^\varepsilon)^n v \to \theta_f(A_f)^n v \text{ weakly in } L^p(\Omega)^3.
\]

\[
\begin{align*}
\varepsilon(t) &= u^0 + tv^0 + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A^\varepsilon)^n u^0 + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n + 1)!} (A^\varepsilon)^n v^0 \\
\to u^0 + tv^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A_f)^n u^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n + 1)!} (A_f)^n v^0 \\
&:= u^H(t).
\end{align*}
\]
Trotter-Kato Theorem and weak convergence of generators of semigroups

For $n = 1, 2, 3, \ldots$, as $\varepsilon \to 0$,

$$(A^\varepsilon)^n v \to \theta_f (A_f)^n v \text{ weakly in } L^p(\Omega)^3.$$
Trotter-Kato Theorem and weak convergence of generators of semigroups

For \( n = 1, 2, 3, \ldots \), as \( \varepsilon \to 0 \),

\[(A^\varepsilon)^n v \to \theta_f (A_f)^n v \text{ weakly in } L^p(\Omega)^3.\]

\[u^\varepsilon(t) = u^0 + tv^0 + \sum_{n=1}^\infty \frac{t^{2n}}{(2n)!} (A^\varepsilon)^n u^0 + \sum_{n=1}^\infty \frac{t^{2n+1}}{(2n+1)!} (A^\varepsilon)^n v^0\]

\[\text{weak} \quad \to \quad u^0 + tv^0 + \theta_f \sum_{n=1}^\infty \frac{t^{2n}}{(2n)!} (A_f)^n u^0 + \theta_f \sum_{n=1}^\infty \frac{t^{2n+1}}{(2n+1)!} (A_f)^n v^0\]

\[:= u^H(t).\]

\[\ddot{u}^H(t) = \theta_f \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} (A_f)^{n+1} u^0 + \theta_f \sum_{n=0}^\infty \frac{t^{2n+1}}{(2n+1)!} (A_f)^{n+1} v^0\]

\[= \theta_f A_f (u^0 + tv^0) + A_f \left( \theta_f \sum_{n=1}^\infty \frac{t^{2n}}{(2n)!} (A_f)^n u^0 + \theta_f \sum_{n=1}^\infty \frac{t^{2n+1}}{(2n+1)!} (A_f)^n v^0 \right)\]
Trotter-Kato Theorem and weak convergence of generators of semigroups

For \( n = 1, 2, 3, \ldots \), as \( \varepsilon \to 0 \),

\[
(A^\varepsilon)^n v \to \theta_f (A_f)^n v \quad \text{weakly in } L^p(\Omega)^3.
\]

\[
u^\varepsilon(t) = u^0 + tv^0 + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A^\varepsilon)^n u^0 + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n + 1)!} (A^\varepsilon)^n v^0
\]

weakly

\[
u^0 + tv^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A_f)^n u^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n + 1)!} (A_f)^n v^0
\]

:= \( u^H(t) \).

\[
\ddot{u}^H(t) = \theta_f \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} (A_f)^{n+1} u^0 + \theta_f \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n + 1)!} (A_f)^{n+1} v^0
\]

\[
= \theta_f A_f (u^0 + tv^0) + A_f \left( \theta_f \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A_f)^n u^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n + 1)!} (A_f)^n v^0 \right)
\]

\[
= \theta_f A_f (u^0 + tv^0) + A_f (u^H(t) - (u^0 + tv^0))
\]
Trotter-Kato Theorem and weak convergence of generators of semigroups

For \( n = 1, 2, 3, \ldots \), as \( \varepsilon \to 0 \),

\[
(A^\varepsilon)^n v \to \theta_f(A_f)^n v \text{ weakly in } L^p(\Omega)^3.
\]

\[
\begin{align*}
\varepsilon(t) &= u^0 + tv^0 + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A^\varepsilon)^n u^0 + \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (A^\varepsilon)^n v^0 \\
\to &= u^0 + tv^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A_f)^n u^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (A_f)^n v^0 \\
&:= u^H(t).
\end{align*}
\]

\[
\begin{align*}
\ddot{u}^H(t) &= \theta_f \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} (A_f)^{n+1} u^0 + \theta_f \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (A_f)^{n+1} v^0 \\
&= \theta_f A_f(u^0 + tv^0) + A_f \left( \theta_f \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} (A_f)^n u^0 + \theta_f \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (A_f)^n v^0 \right) \\
&= A_f u^H(t) + (\theta_f - 1) A_f(u^0 + tv^0)
\end{align*}
\]
Future Work

- Numerical simulations.

- Multiscale analysis for nonlinear peridynamics.
Thank You