The Mathematics Behind Biological Invasions
Project Ideas

1. Read the papers by Andow *et al* (1990) and Lubina and Levin (1988). Calculate the spread rate for a population with exponential growth, diffusion and advection (drift term) which obeys

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} + ru$$

   (HINT: spread will now be different in the x and $-x$ directions.) To what biological situation in the above papers would this model apply? State whether Andow *et al* and whether Lubina and Levin calculated parameters independently from the spread they were trying to explain with the model. Support your assertions.

2. Solve

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u(1-u) \qquad u(x,0) = u_0(x)$$

   on a domain $0 \leq x \leq 200$ with zero flux boundary conditions. For $u_c = 0.01$ plot $x_c(t)$ versus t for $u_0(x) = 1 - H(x-1)$ where $H(\cdot)$ is the Heaviside step function, and for $u_0(x) = \exp(-x)$. Comment on the solutions in view of the theory from class.

3. Use Maple to solve the traveling wave equation

$$U'' + cU' + U(1-U) = 0$$

   numerically in the $U - U'$ phase plane for (i) $c = 3$, (ii) $c = 2$, (iii) $c = 0.5$. Plot the corresponding solutions $U(z)$ versus z. Comment on how the numerical solutions tie in with our analytical results.

4. Read pages 289–291 of Murray (1989), where the solution to

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial}{\partial x}(u\frac{\partial u}{\partial x})$$

   (logistic growth and nonlinear diffusion) is discussed. Write up your solution to the traveling wave problem based on this discussion. Be sure to include all details. Use Maple to find the solution in the $U - U'$ phase plane for the case $c = 1/\sqrt{2}$. How does this differ from the solution to Fisher's equation?

5. Consider the scalar reaction diffusion with an Allee effect (negative growth rates at low density)

$$\frac{\partial u}{\partial t} = u(1-u)(u-a) + \frac{\partial^2 u}{\partial x^2}.$$

   (a) Draw $f(u)$ and indicate the flow of $\dot{u} = f(u)$. Look for a traveling wave solution $u(x,t) = U(z)$, $z = x - ct$, where c is the traveling wave velocity. Transform the above equation to an ODE for the traveling wave. Impose reasonable boundary conditions as $z \to \pm\infty$.

(b) Linearize about the relevant equilibria in the $U - U'$ phase plane. Calculate the eigenvalues and corresponding eigenvectors. Draw the heteroclinic orbit between the relevant equilibria that defines the traveling wave solution. Be sure to make sure that the orbit is tangent to the eigenvectors at the equilbria.

(c) Consider the case where $U' = AU(1-U)$ and thus the orbit is defined by a parabola in the phase space. Calculate U'' and plug the expressions for U' and U'' into the traveling wave equation. Use this to show that the parabola describes a solution with unique wave velocity $c = c^* = \sqrt{2}(1/2 - a)$ and unique value for A which you should find. Integrate $U' = AU(1-U)$ with respect to z and apply boundary conditions to give the traveling wave solution $U(z)$.

(d) Using the c-dependence (velocity dependence) of the eigenvectors in the phase plane, discuss the qualitative behavior of trajectories for $c < c^*$ and $c > c^*$. Why is it reasonable to expect a unique wave speed c^* for the traveling wave problem?

6. Prove that a scalar (single species) autonomous reaction-diffusion model has no non-trivial homoclinic traveling wave solutions with non-zero wave speed.

7. Let $k(x) = N(0, 2D)$. Show that

$$N_{t+1} = \int_{-\infty}^{\infty} R_0 k(x-y) N_t(y)\, dy, \qquad N_0(x) = \delta(x)$$

satisfies the Malthusian growth and dispersal equation

$$\frac{\partial N}{\partial t} - D\frac{\partial^2 N}{\partial x^2} + rN$$

at times $t = 0, 1, 2, \ldots$, providing R_0 is defined appropriately in terms of r. Show that

$$c^* = \min_{s>0}\left\{\frac{1}{s}\log(R_0 M(s))\right\} = 2\sqrt{\log(R_0)D}.$$

where $M(s)$ is the moment generating function for the kernel $k(x)$. Compare with the reaction diffusion case.

8. Consider the asymptotic spread rate

$$c^* = \min_{s>0}\left\{\frac{1}{s}\log(R_0 M(s))\right\}.$$

for the case

$$k(x) = (1-p)k_1(x) + pk_2(x),$$

where

$$k_i(x) = \frac{\alpha_1}{2}\exp(-|\alpha_i x|),$$

and $0 \le p \ll 1$, $\alpha_1 \gg \alpha_2 > 0$. Choose $\alpha_1 = 10$, $\alpha_2 = 1$ and evaluate c^* numerically (say, using Maple) for the cases $p = 0$ and $p = 0.01$. Comment on the different values of c^*. Explain the difference with the aid of a diagram of $\exp(sc)$ versus $R_0 M(s)$ for the cases $p = 0$ and $p = 0.01$.

9. A spatial contact model for epidemics, analysed by Mollison (1972, 1977) is

$$\frac{\partial n}{\partial t} = r\bar{n}, \qquad \alpha > 0,$$

where

$$\bar{n} = \int_{-\infty}^{\infty} n(y,t)k(x-y)\,dy.$$

and $n(x,t)$ is population density. When density dependent effect is taken into account the equation is extended to

$$\frac{\partial n}{\partial t} = \alpha\bar{n}\left(1 - \frac{n}{K}\right)$$

(a) Explain the terms in the density-dependent equation using words.

(b) Look for a traveling wave solution for the linear problem of the form $n(x,t) = A\exp(-s(x-ct))$ and calculate a dispersion relation between the wave speed c and the steepness of the wave.

(c) Write a formula for the minimum wave speed c^*. Show how to find the minimum wave speed graphically for the case $k(x) = \alpha\exp(-\alpha|x|)/2$. (HINT: consider a graphical depiction analogous to the one in the above problem.)