Math 5110
Old midterm
October 3, 2017

Math 5110 Old Midterm

Do all four problems (25 points each), Write readable answers on the test, but feel free to use or hand in additional paper if necessary. One page of notes, no calculators, no cheating.

1. Loud rumbling noises can be created by the successive breakdown of mechanical components. Suppose a building has two noisy mechanical components, Rumblers and Grumblers, with R_t and G_t respectively representing the number of broken ones in week t. The vibrations they create lead to further breakdowns. Furthermore,

- half (50%) of the broken Rumblers are fixed each week;
- each broken Rumbler generates one new broken Grumbler;
- each broken Rumbler generates no new broken Rumblers;
- one-fourth (25%) of the broken Grumblers are fixed each week;
- each broken Grumbler creates b broken Rumblers;
- each broken Grumbler generates no new broken Grumblers.

(a) Write these rules as a matrix equation.

(b) What is the largest eigenvalue when $b = 0$?

(c) For what value of b will the number of broken Rumblers remain constant in the long run? What is the limit of the ratio of broken Rumblers to broken Grumblers?

(d) For what value of b will the number of broken Rumblers and Grumblers be equal in the long run? How fast would the noise be increasing?

(a) We begin by considering how R_t changes with each discrete time step. From the information above, broken Rumblers are added to the population at a rate bG_t, and removed at a rate $\frac{1}{2}R_t$, corresponding to

$$R_{t+1} = bG_t + \tfrac{1}{2}R_t.$$

Similarly, broken Grumblers are added to the population at a rate R_t and removed at a rate $\frac{1}{4}G_t$, corresponding to

$$G_{t+1} = R_t + \tfrac{3}{4}G_t.$$

When written in matrix form, these two equations become

$$\begin{bmatrix} R_{t+1} \\ G_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & b \\ 1 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} R_t \\ G_t \end{bmatrix}.$$

(Observe that the 3rd and 6th pieces of information had to be included, but if they were nonzero, they would only lower the effective repair rate.)

(b) When $b = 0$, the matrix is lower triangular and the eigenvalues can be read off the main diagonal. The dominant eigenvalue is the one with the highest absolute value, $\lambda = \frac{3}{4}$.

(c) If we solve the first equation for equilibrium R^* value,

$$R^* = bG^* + \frac{1}{2}R^*,$$

we obtain $R^* = 2bG^*$. Hence

$$G^* = 2bG^* + \frac{3}{4}G^*$$

is solved by $0 = \left(2b - \frac{1}{4}\right)G^*$. (Note: This is inadequate. You MUST show your work on a test.) Hence either $R^* = G^* = 0$, corresponding to an unstable trivial equilibrium, or $b = \frac{1}{8}$, corresponding to a stable locus of equilibria. Thus $b = \frac{1}{8}$ satisfies the parameters of the problem.

Furthermore, to find the ratio $\frac{R^*}{G^*}$, we simplify the second equation in the model to

$$\begin{aligned} G^* &= R^* + \frac{3}{4}G^* \\ \frac{R^*}{G^*} &= \frac{1}{4}. \end{aligned}$$

(d) To make the number of Rumblers and Grumblers equal in the long run, we set $R_t = G_t$ and find b such that $R_{t+1} - R_t = G_{t+1} - G_t$:

$$\begin{aligned} bG_t + \tfrac{1}{2}G_t - G_t &= G_t + \tfrac{3}{4}G_t - G_t \\ \left(b + \frac{1}{2} - 1\right)G_t &= \tfrac{1}{4}G_t \\ b &= \frac{5}{4}. \end{aligned}$$

I don't think there's enough information to find the noise, because we don't know how much noise each Rumbler or Grumbler is producing. We *could* find $(R_{t+1} + G_{t+1}) - (R_t + G_t) = \frac{3}{2}G^*$, so the total number of machines increases by 50% every week.

♥

2. Suppose the aggravation level A_t follows the discrete time dynamical system shown in the figure (the thick curve), the graph of $A_{t+1} = g(A_t)$, where $g'(0) = -22.5$. The thin line shows where $A_{t+1} = A_t$.

(a) Cobweb starting from an aggravation level of 0.2.

(b) Describe the equilibria of this system.

(c) Suppose the system were generalized to $A_{t+1} = g(A_t) + c$. What bifurcation is occurring at $c = 0$ (the case shown)?

(d) What bifurcation will occur with $c > 0$?

(e) What bifurcation will occur with $c < 0$?

(f) Draw a bifurcation diagram for this system as we vary the parameter c.

(a) We can draw this in class.

(b) The equilibrium point at about $A_t = 0.4$ is stable, and the equilibrium point at about 1.3 is semistable (assuming the curve is tangent to $A_{t_1} = A_t$ at this point). To clarify, the system tends toward the higher equilibrium point if it is approached from above, but tends away from the higher equilibrium point, towards the lower one, if the higher equilibrium point is approached from slightly below.

(c) At the point where the curve is tangent to $A_{t_1} = A_t$, two equilibrium points are being created. We call this a saddle-node bifurcation (although the name doesn't quite make sense in discrete-time systems).

(d) Recall that raising c will translate the graph upwards. As $c > 0$, there will be a point where the lower rung of the graph is tangent to $A_{t_1} = A_t$, corresponding to a second saddle-node bifurcation.

(e) As $c < 0$, the graph will translate downwards, such that the slope of the line intersecting with $A_{t_1} = A_t$ is less than -1, destabilizing the existing equilibrium point. This is a flip bifurcation.

(f) We can draw this in class.

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3. Constant low rumbling noises can drive people into a homicidal rage, and seeing others rage can be catching. In particular, suppose the number of people in a homicidal rage in a week t is H_t and it obeys the discrete-time dynamical system

$$H_{t+1} = \tfrac{3}{4}H_t^p + 3 - 2p$$

where p can differ among students.

(a) Suppose first that $p = 1$ for all students. Find the equilibrium number of homicidal ragers and prove that it is stable.

(b) Now suppose that half of the students have $p = 1$ and the other half have $p = 2$. Write the discrete-time dynamical system for this case (combine the system from (a) with the system you get by setting $p = 2$).

(c) Find the equilibria of this new system and their stability.

(d) How would you model this if the values of p were distributed uniformly over all values of p between 1 and 2?

(a) Set $p = 1$. Equilibria are found by setting $H^* = \frac{3}{4}H^* + 1$, which is solved by $H^* = 4$. The eigenvalue of the linearized system is $\frac{3}{4} < 1$, so this system is stable.

(b) Call the second group of students M_t, for murderers. Also, suppose that M_t and H_t both count as raging people, so the total number of raging students is $H_t + M_t$. The new system is then

$$\begin{aligned} H_{t+1} &= \tfrac{3}{4}(H_t + M_t) + 1 \\ M_{t+1} &= \tfrac{3}{4}(H_t + M_t)^2 - 1. \end{aligned}$$

Suppose we only want to track the total number of killer students, $K_t = H_t + M_t$. By adding these two equations together, we obtain

$$K_{t+1} = \frac{3}{4}(K_t + K_t^2),$$

and analyze this new system.

(c) The above system has equilibria that satisfy

$$K^* = \frac{3}{4}(K^* + K^{*2}),$$

which is solved by $K^* = 0$ and $K^* = \frac{1}{3}$. To classify these, we linearize the syste. About 0, the linearized system is

$$K_{t+1} = \left(\frac{3}{4} + 2(0)\right)K_t,$$

which has an eigenvalue less than one, so this equilibrium is stable. Similarly, the linearization about $K^* = \frac{1}{3}$ is

$$K_{t+1} = \left(\frac{3}{4} + \frac{2}{3}\right)K_t,$$

which has an eigenvalue greater than one, hence is unstable.

(d) I would take the average over all values of p to obtain the system

$$H_{t+1} = \tfrac{3}{4} H_t^{3/2}.$$

♥

4. It is hypothesized that complaining to the authorities can lead to the repair of Rumblers, Grumblers and other noise-producing machinery. Let M_t represent the total number of broken machines in week t and C_t the number of complaints. Suppose that

$$M_{t+1} = \frac{\lambda M_t}{1+C_t} C_{t+1} \qquad\qquad = rM_t.$$

(a) Explain these equations.

(b) Find the equilibria. What are the conditions on the parameters for a positive equilibrium to exist?

(c) Find the stability of any positive equilibria.

(d) What do you plan to do tomorrow?

(a) The second equation assumes a complaint rate which is proportional to the number of broken machines, but is independent of other complaints (which makes sense). For the first equation, λ represents the maximal rate of machines causing each other to break. As the number of complaints increases, the rate of machine breakage asymptotically approaches zero. Furthermore, suppose $r > 0$, since this will be useful in later parts of the problem.

(b) Set

$$\begin{aligned} M^* &= \frac{\lambda M^*}{1+C^*} \\ C^* &= rM^*. \end{aligned}$$

The second equation can be substituted into the first, providing the quadratic equation $rM^{*2} + (1-\lambda)M^*$ (again, SHOW YOUR WORK ON THE TEST). As a result of solving this quadratic, we obtain two equilibria for (M^*, C^*) : (0,0) and $\left(\frac{\lambda-1}{r}, \lambda - 1\right)$. If $r > 0$, the second equilibrium is positive provided $\lambda > 1$.

(c) This system cannot be written as a matrix equation, but it can be linearized. The linearized system about positive equilibrium (M^*, C^*) is

$$\begin{aligned} \begin{bmatrix} M_{t+1} \\ C_{t+1} \end{bmatrix} &= \begin{bmatrix} \frac{\lambda}{1+C^*} & -\frac{\lambda M^*}{(1+C^*)^2} \\ r & 0 \end{bmatrix} \begin{bmatrix} M_t \\ C_t \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{\lambda-1}{r\lambda} \\ r & 0 \end{bmatrix} \begin{bmatrix} M_t \\ C_t \end{bmatrix}. \end{aligned}$$

This matrix has eigenvalues $\mu_\pm = \frac{\lambda \pm \sqrt{-4\lambda+5\lambda^2}}{2\lambda}$, which are both less than one when

$$\begin{aligned} 1 &> \frac{\lambda + \sqrt{-4\lambda+5\lambda^2}}{2\lambda} \\ \lambda &> \sqrt{-4\lambda+5\lambda^2} \\ -4\lambda^2 + 4\lambda &> 0 \\ 4\lambda(1-\lambda) &> 0. \end{aligned}$$

We already required $\lambda > 1$, so this condition is satisfied for all positive equilibria. We conclude that the positive equilibrium is always stable, if it exists. There will always be broken machines!

(d) Probably lead a review session and then do math all day.

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