Math 5110: Homework Assignment 9
Due November 7, 2017

1. The classical Lotka-Volterra model of competition assumes that each species reduces the per-capita growth rate of both itself and the other species linearly. Consider an alternative model where

$$\begin{aligned}\frac{dN_1}{dt} &= r_1N_1(1 - a_{11}^2N_1^2 - a_{12}N_2)\\ \frac{dN_2}{dt} &= r_2N_2(1 - a_{22}^2N_2^2 - a_{21}N_1).\end{aligned}$$

Here r_1 and r_2 are the maximum growth rates of species 1 and 2 respectively, and a_{ij} gives the effect of species j on species i.

   a. Find the conditions under which N_1 increases from the equilibrium where N_2 is positive and $N_1 = 0$.

   b. Find the conditions for coexistence.

   c. Draw the phase-plane in this case.

2. Coexistence is impossible in general for the special case of the model

$$\begin{aligned}\frac{dN_1}{dt} &= f_1(R)N_1 - \mu_1N_1\\ \frac{dN_2}{dt} &= f_2(R)N_2 - \mu_2N_2\\ 0 &= g(N_1, N_2, R)\end{aligned}$$

with $\mu_1 = \mu_2 = \mu$. Does the more general version where μ_1 and μ_2 can take on different values have the same result? In this system, we assume that R can be found by solving an algebraic equation rather than a differential equation. Explicitly find the equilibria with the form

$$f_i(R) = \frac{\alpha_iR}{K_i + R}$$

and $g(N_1, N_2, R) = S - \alpha_1N_1R - \alpha_2N_2R$. Draw the nullclines and equilibria for a case where species 1 wins.

3. Consider the following model of two competing diseases.

$$\begin{aligned}\frac{dS}{dt} &= -c_1I_1S - c_2I_2S + \gamma_1I_1 + \gamma_2I_2\\ \frac{dI_1}{dt} &= c_1I_1S - \gamma_1I_1 - \delta I_1I_2\\ \frac{dI_2}{dt} &= c_2I_2S - \gamma_2I_2 + \delta I_1I_2\end{aligned}$$

Suppose that S, I_1 and I_2 represent the fractions of individuals who are susceptible, infected with disease 1, and infected with disease 2 respectively.

   a. Figure out what all the parameters mean, with particular attention to δ.

   b. Use the fact that $S + I_1 + I_2 = 1$ to write a system of two differential equations for I_1 and I_2.

   c. Assume that $\delta = 0$. Show that only one disease will persist. Find the conditions under which disease 1 wins.

   d. Pick some simple values of the parameters for which disease 1 wins when $\delta = 0$. Is there a positive value of δ for which the two diseases coexist at a stable equilibrium? Can you explain this in words a biologist could understand?