Math 5110: Homework Assignment 3
Due September 12, 2017

1. Consider a discrete-time model of the concentration of medication in the bloodstream, M_t,

$$M_{t+1} = M_t - f(M_t)M_t + S,$$

where $f(M_t)$ is the fraction absorbed and S is the daily dosage. Suppose that

$$f(M) = \frac{M^n}{K^n + M^n}.$$

Set $K = 2$ and $S = 1$. Show that the equilibrium level is $M^* = 2$ for any value of n. For which values of n will solutions oscillate toward the equilibrium? Are there any values of n for which the equilibrium is unstable? Sketch a cobweb diagram of an illuminating case.

2. Suppose a cancer is growing according to

$$C_{t+1} = (m\sigma + p)C_t$$

where p is the decreasing function of C_t

$$p(C) = \frac{C}{1 + C^2}.$$

Suppose that $m = 2$ because cells are dividing. We will study how the behavior depends on the value of σ.

   a. Graph $m\sigma + p(C)$ as a function of C for $\sigma = 1/8, 1/4, 1/2$ and $3/4$. Can you tell when the cancer will grow?

   b. Draw cobweb diagrams for each of these values of σ.

   c. What are the stable and unstable equilibria in each case?

   d. Draw the bifurcation diagram. What happens at $\sigma = 1/2$? Is something wrong with the model?

3. Newton's method for numerically solving equations can be formalized as a discrete-time dynamical system. For example, to find a numerical value of $\sqrt{3}$, we must solve $x^2 = 3$. Think of this as solving the equation $f(x) = 0$ where $f(x) = x^2 - 3$. Newton's method consists of picking a guess (like $x_0 = 2$), and replacing the equation $f(x) = 0$ with the linear equation $\hat{f}(x) = 0$ where $\hat{f}(x)$ is the tangent line to $f(x)$ at the first guess x_0.

   a. Graph $f(x)$ and $\hat{f}(x)$ at the guess $x_0 = 2$.

   b. Solve $\hat{f}(x)$ to find x_1.

   c. Find the tangent line at x_1, and use it to find the next guess x_2.

   d. Write down the discrete-time dynamical system that gets from one guess to the next. Graph the updating function.

   e. Find the equilibrium and its stability.

   f. Newton's method converges to the correct solution with astonishing speed. Do you have any idea why?