Math 5110: Homework Assignment 13
Due December 5, 2017

1. Consider the nondimensional predator-prey system

$$\frac{dx}{dt} = x[x(1-x) - y]$$
$$\frac{dy}{dt} = y(x - a),$$

where $x \geq 0$ is the dimensionless population of the prey, $y \geq 0$ is the dimensionless population of the predator, and $a \geq 0$ is the death rate of predators.

   a. Sketch the nullclines in the first quadrant ($x,\ y \geq 0$).

   b. Find the equilibria and their stability.

   c. Sketch the phase portrait for $a > 1$ and show that the predators go extinct.

   d. Show that a Hopf bifurcation occurs at $a_c = \frac{1}{2}$.

   e. Sketch all the different possible phase portraits with $0 < a < 1$.

   f. Explain, in words a biologist could understand, why changing a produces oscillations.

2. Suppose that individuals reproduce at per capita rate r but die at age T.

   a. Write a delay differential equation for this population.

   b. Assume that the population grows according to $N(t) = e^{\lambda t}$. Write the equation that λ must satisfy.

   c. Find a necessary and sufficient condition on r and T for there to be a positive solution for λ. Explain this condition (what is required for a population to replace itself?).

3. Consider a population following the equation

$$\frac{dN}{dt} = rN - \mu N. \tag{1}$$

Individuals reproduce at per capita rate r and die at per capita rate μ. Suppose that $\mu = 1$.

   a. Find and graph the growth rate of the population as a function of r.

   b. Find the average time that an individual remains alive (if you can't figure out how to do this, try guessing).

   c. We can now compare a population where individuals remain alive for time T **on average** (equation 1) with one where individuals remain alive for time T **exactly** (exercise **2**). Use the average you found in part **b** as T, and numerically solve for the growth rate λ (found in exercise **2**) for five interesting values of r. Make sure to use at least one value of r which produces a small growth rate.

   d. Plot these values on your graph from part **a**. Do they match? How do they differ? Can you explain the difference?