

NAME: _____

MATH 5110
The Final

Do all problems. Feel free to use or hand in additional paper if necessary. Make sure I can find both the answer and how you got it. Two pages of notes, no calculators, cell phones etc.

The theme of this test is diabetes. There are three key state variables, blood glucose level G , blood insulin level I , and number of β cells. The β cells release insulin in response to blood glucose after eating, and this insulin induces muscle cells to take up that sugar. Diabetics have high blood sugar levels either because muscle cells ignore the insulin signal and fail to take up glucose, or because β cells fail to make enough insulin.

SHORTER PROBLEMS (30 points each, take about 15 minutes each).

- A.** A very small animal, *Caenorhabditis insulans*, has at most two β cells, but could have zero or one. Suppose that these cells reproduce at a per capita rate of $r(2 - \beta)$ when there are β of them, that they die at per capita rate θG^2 when the glucose level is G (assumed constant here). If there are fewer than two β cells, new cells are generated from stem cells at rate S .
- a.** If an animal has zero cells at some time, how long on average will it remain with zero cells?
 - b.** Write down differential equations for probabilities describing the different possible number of β cells.
 - c.** What is the equilibrium of this system?

- B.** This problem concerns only the effects of eating. Suppose there is a feedback between how much a person eats on one day and how much they ate the previous day, leading to a discrete-time dynamical system for blood glucose

$$G_{t+1} = 2 - 4c + cG_t^2$$

where c is a positive parameter that might differ between different people.

- a.** What kind of feedback is this?
- b.** Show that $G^* = 2$ is an equilibrium for any value.
- c.** For which values of c is this equilibrium stable?
- d.** Find and identify all bifurcations that occur for $c > 0$.
- e.** Draw a bifurcation diagram.

C. This problem too models only eating. Suppose blood glucose follows the differential equation

$$\frac{dG}{dt} = \frac{4k}{1 + G/2} - kG$$

for some positive value of k .

- a. Find the equilibrium (possibly the same as in **1**) and its stability.
- b. Find an approximate solution near the equilibrium.
- c. Draw the phase-line.
- d. What does the parameter k change about the system? Why are there no bifurcations?

D. Consider the following system of differential equations

$$\begin{aligned}\frac{dG}{dt} &= \sigma - \eta GI - \delta_G G \\ \frac{dI}{dt} &= \alpha\beta G - \theta\eta GI - \delta_I I \\ \frac{d\beta}{dt} &= \epsilon(k\theta\beta(2 - \beta) - \theta G^2\beta)\end{aligned}$$

- a.** Explain the non-linear terms in these equations.
- b.** Suppose ϵ is small. Which equation or equations could you put into quasi-steady state?
- c.** Explain how you would use this to find a lower dimensional system.
- d.** What would this lower dimensional system be useful for?

LONGER PROBLEMS (40 points each, take about 25 minutes each).

1. Suppose that glucose and insulin follow the differential equations

$$\begin{aligned}\frac{dG}{dt} &= \sigma - \eta GI - \delta_G G \\ \frac{dI}{dt} &= \alpha\beta G - \theta\eta GI - \delta_I I\end{aligned}$$

where β has been assumed to be constant.

- a. Find the nullclines and draw direction arrows.
- b. Figure out how many equilibria there are.
- c. Find the stability of the equilibria.
- d. What do you know about the eigenvalues of this system?
- e. What happens to the equilibria as σ (the rate of glucose intake) increases?

2. A new treatment has been developed that can take pre-diabetic people back to a healthy state before it is too late. Let P_t be the number of pre-diabetic people in year t and T_t the investment in distribution of the treatment (in millions of dollars). Suppose they follow the discrete-time system

$$\begin{aligned}P_{t+1} &= 1500 - \frac{T_t}{1 + T_t}P_t \\T_{t+1} &= 0.9T_t + \frac{P_t}{10^4}.\end{aligned}$$

- a. Explain the terms in these equations.
- b. This has one equilibrium with $T^* = 1$. Find the associated equilibrium value of P .
- c. Find the stability of the equilibrium.
- d. What do you know about the eigenvalues and what do they tell you about the behavior of solutions near the equilibrium?