NAME:____________________

MATH 5110
The Final

Do all problems. Feel free to use or hand in additional paper if necessary. Make sure I can find both the answer and how you got it. Open book and notes, no calculators, cell phones etc.

SHORT PROBLEMS (10 points each, take no more than 5 minutes each).

A. Suppose that N_t and P_t are populations of hosts and parasitoids respectively in year t. Both species live at most 1 year. A fraction $f = \dfrac{1}{1+P_t^2}$ of hosts survive parasitoid attacks, and each of these survivors has $\dfrac{\lambda}{1+N_t}$ offspring that survive until year $t+1$. Each host that does not survive produces exactly c new parasitoids that survive until year $t+1$. Write a discrete time dynamical system describing these two populations? How do they differ from the Nicholson-Bailey equation?

B. What is the difference between an excitable system and a molecular clock? Describe the equilibria and their stability in each case.

C. A species has three types of individuals, wee babies (age 1), juveniles (age 2) and adults (ages 3 and greater). All are female. Wee babies have a 0.5 chance of surviving to the juvenile state, juveniles have a 0.5 chance of surviving to adulthood, and adults have a 0.4 chance of surviving (and thus remaining as adults). Adults produce 2.0 wee babies. Write equations that describe this population. How would you figure out whether the population would grow?

D. An enzyme E complexes with a substrate S. The resulting complex C can break down into two molecules of the enzyme. All reactions are reversible. Write differential equations describing this system. What quantity is conserved?

LONG PROBLEMS (40 points each, take about 25 minutes each).

1. A team of modelers led by V. Guaraldi developed a three dimensional system of ordinary differential equations to describe the dynamics of Streptococcus (strep) bacteria, S, immune cells that attack and destroy them, I, and a cytokine C (a chemical important in regulating the immune response).

$$\begin{aligned}\frac{dS}{dt} &= \frac{\rho S}{k_1 + S} - \alpha I S \\ \frac{dI}{dt} &= \frac{\beta C^2}{k_2^2 + C^2} I - \alpha p I S - \delta I \\ \frac{dC}{dt} &= \gamma I S - \mu C\end{aligned}$$

All parameters are assumed to be non-negative.

a. How do immune cells affect strep? Do they alter birth or death rates?

b. Describe the per capita reproduction of strep in the absence of immune cells, and give a reason why it might follow the given form.

c. What does that C^2 term mean and why might it be there? What does k_2 mean?

d. What does the parameter p mean? What is a reasonable range of values and why?

e. When are cytokines generated? How are they degraded?

f. Suppose that cytokine dynamics are fast relative to other processes in the model. What parameters will be large? Use this to reduce this model to a two dimensional system.

2. N. K. Cole et al derived a similar but subtly different two-dimensional model of the interaction between strep S and immune cells I given by

$$\begin{aligned}\frac{dS}{dt} &= \frac{\rho S}{k_1 + S} - \alpha I S \\ \frac{dI}{dt} &= \frac{\beta S}{k_2 + S} I - \alpha p I S - \delta I\end{aligned}$$

All parameters are assumed to be non-negative.

a. What is different about this model?

b. Find the equilibria and nullclines. Draw the phase plane, complete with direction arrows, in the case with the most possible equilibria.

c. Find the matrix needed to determine the stability of the equilibria. Determine the sign (positive or negative) for all the easy terms in the matrix, and use your phase plane diagram to deduce the signs of the others.

d. Do the same in the case with the fewest equilibria. What would happen to the infection in the long run? Explain why large values of some parameters and small values of others tend to create this case.

3. S. Roller, in a long-term collaboration with Ö. Mannheim, published a long series of discrete-time models, the simplest of which is

$$\begin{aligned} S_{t+1} &= \left(\frac{\rho S_t}{k_1 + S_t}\right)\left(\frac{k_3}{k_3 + I_t}\right) \\ I_{t+1} &= \frac{\beta S_t}{k_2 + S_t} I_t. \end{aligned}$$

a. Explain how the per capita reproduction of S and I each depend on S_t and I_t.

b. What would happen to S in the absence of I? Draw a diagram to illustrate this result.

c. Find the equilibria of the two-dimensional system. What are the conditions for existence of an equilibrium with $I = 0$ and $S > 0$? What are the conditions for existence of an equilibrium with $I > 0$ and $S > 0$?

d. Assuming that a positive equilibrium exists, what is the only way it can become unstable? What kind of bifurcation would occur if it did?