NAME:____________________

MATH 5110
The Final

Do all five problems. Write readable answers on the test, but feel free to use or hand in additional paper if necessary. **Make sure I can find both the answer and how you got it**. Open book and notes, no calculators. 50 points each.

1. Suppose two proteins inhibit each other's production, but function on a daily time scale according to the discrete-time dynamical system

$$\begin{aligned} x_{t+1} &= f(y_t) + \sigma_1 x_t \\ y_{t+1} &= g(x_t) + \sigma_2 y_t \end{aligned}$$

where x and y represent the concentrations of the two proteins, f and g are positive decreasing functions, and σ_1 and σ_2 are the fraction of x and y respectively that persist until the next day. Find the condition on $f'(y^*)$ and $g'(x^*)$ that makes an equilibrium (x^*, y^*) unstable, and explain what it means graphically. Does the existence of an unstable equilibrium guarantee that this system works as a switch?

2. Suppose two organisms, with population sizes N_1 and N_2, are competing for the resource R according to the system

$$\begin{aligned} \frac{dN_1}{dt} &= \frac{r_1 N_1 R}{1 + \alpha_1 R} - \delta_1 N_1 \\ \frac{dN_2}{dt} &= \frac{r_2 N_2 R}{1 + \alpha_2 R} - \delta_2 N_2 \\ \frac{dR}{dt} &= S - r_1 N_1 R - r_2 N_2 R - bR. \end{aligned}$$

a. Describe (in 15 words or fewer) how the per capita growth of the two species depends on R.

b. Scale R by its equilibrium value in the absence of the two organisms (if $N_1 = N_2 = 0$).

c. Scale time in all three equations using the rate constant in the scaled R equation.

d. Suppose that r_1, r_2, δ_1, and δ_2 are all of similar magnitude, but that b can differ. For what values of b are resources in quasi-steady state? Write and label the equations associated with the inner and outer solutions in this case, and find the inner solution.

3. Consider a model of prey with population size N and predators with population size P

$$\begin{aligned} \frac{dN}{dt} &= rN(1-\frac{N}{K}) - aNf(P)P \\ \frac{dP}{dt} &= caNf(P)P - \delta P \end{aligned}$$

where $f(P)$ is an increasing, bounded function with $f(0) = 0$.

- **a.** What does the inclusion of $f(P)$ mean (in 10 words or fewer)?
- **b.** What bifurcations can this system have?
- **c.** Draw the phase-plane, complete with direction arrows, in the case with the most equilibria.

4. Consider an autocatalytic enzyme that obeys

$$A + X \xrightarrow{k_1} X + X \xrightarrow{k_2} \text{crud}$$

After the reaction, molecules of X are monomers (singletons) and decay independently at rate k_2. Assume that A is held constant (a constant supply of substrate is always available). Suppose that k_1 is a function of X with the form

$$k_1(X) = \frac{k_0}{h(X)}$$

- **a.** Suppose first that $h(X)$ is an increasing function. Describe all possible behaviors of the differential equation for X and the bifurcations that occur as a function of A.
- **b.** What other form (other than increasing) of the function $h(X)$ is needed for this system to be able to function as a switch? What additional bifurcations occur in this case?